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Bivariate C^1 Cubic Spline Space Over a Nonuniform Type-2 Triangulation and its Subspaces with Boundary Conditions

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Abstract—In this paper, we discuss the algebraic structure of bivariate C^1 cubic spline spaces over nonuniform type-2 triangulation and its subspaces with boundary conditions. The dimensions of these spaces are determined and their local support bases are constructed. © 2005 Elsevier Ltd All rights reserved.

Keywords—Bivariate C^1 cubic spline space, Boundary conditions, Dimension and basis, Nonuniform type-2 triangulation

1. INTRODUCTION

Let $\Omega = [0, x_m] \otimes [0, y_n]$ be a rectangle. For $0 = x_0 < x_1 < \cdots < x_m$ and $0 = y_0 < y_1 < \cdots < y_n$, Ω is divided into mn small rectangles $\Omega_{ij} = [x_i, x_{i+1}] \otimes [y_j, y_{j+1}]$, $i = 0, 1, \ldots, m-1$, $j = 0, 1, \ldots, n-1$, by mesh lines,

$$x = x_i, \quad i = 1, \ldots, m-1, \qquad \text{and} \qquad y = y_j, \quad j = 1, \ldots, n-1.$$

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The triangulation generated by drawing all northeast diagonals in all small rectangles is called a nonuniform type-1 triangulation and is denoted by $\bar{\Delta}_{mn}^{(1)}$. The triangulation generated by drawing all northeast and northwest diagonals in all small rectangles is called a nonuniform type-2 triangulation and is denoted by $\bar{\Delta}_{mn}^{(2)}$ (see Figure 1). Let $h_i = x_i - x_{i-1}$, $t_j = y_j - y_{j-1}$ for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. If $h_i \equiv h$, $i = 1, \dots, m$, and $t_j \equiv t$, $j = 1, \dots, n$, then $\bar{\Delta}_{mn}^{(1)}$ and $\bar{\Delta}_{mn}^{(2)}$ are called (uniform) type-1 triangulation and (uniform) type-2 triangulation, and denoted by $\Delta_{mn}^{(1)}$ and $\Delta_{mn}^{(2)}$, respectively.

For $0 \le r \le k-1$, where r and k are integers, we define $S_k^r(\bar{\Delta}_{mn}^{(i)})$ to be the vector space consisting of all the functions $s(x, y)$ satisfying the following conditions,

(i) $s \in C^r(\Omega)$,

(ii) the restriction of $s(x, y)$ to each triangular element of $\bar{\Delta}_{mn}^{(i)}$ belongs to π_k, where

$$\pi_k = \left\{ \sum_{0 \le i+j \le k} \lambda_{ij} x^i y^j : \lambda_{ij} \in \mathbb{R} \right\}$$

is the linear space of polynomials with total degree at most k.

$S_k^r(\bar{\Delta}_{mn}^{(i)})$ is called a bivariate spline space over $\bar{\Delta}_{mn}^{(i)}$ with degree k and smoothness order r.

Let d be an integer with $0 \le d \le r$. We define

$$S_k^{r,d}\left(\bar{\Delta}_{mn}^{(i)}\right) = \left\{ s \in S_k^r\left(\bar{\Delta}_{mn}^{(i)}\right) : \left.\frac{\partial^\mu s}{\partial \mathbf{n}^\mu}\right|_{\partial\Omega} \equiv 0,\ \mu = 0, \dots, d \right\}, \qquad i = 1, 2, \tag{1}$$

where $\partial\Omega$ is the boundary of Ω and $\mathbf{n}$ is the outward normal unit vector along $\partial\Omega$. Clearly, $S_k^{r,d}(\bar{\Delta}_{mn}^{(i)})$ is a subspace of $S_k^r(\bar{\Delta}_{mn}^{(i)})$, and it is called a bivariate spline space with boundary conditions.

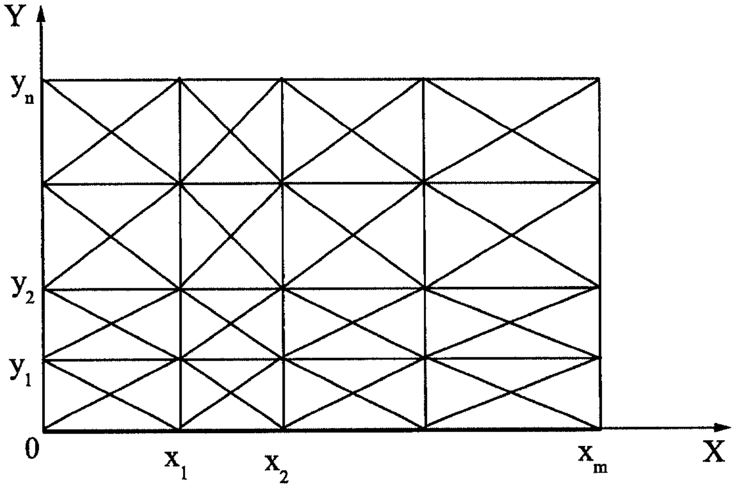


Figure 1 The nonuniform type-2 triangulation $\bar{\Delta}_{mn}^{(2)}$.

Spline spaces and their subspaces with boundary conditions have many applications in mechanical computation, surface fitting, computer aided geometric design (CAGD), finite-element method, and numerical solution of boundary value problem of partial differential equations. In the study of spline spaces, a basic but important problem is to determine the dimension and construct a locally supported basis for the spaces. In this direction, Chui and Schumaker [1] first studied the spline space over a rectangular partition in 1982. For uniform type-1 triangulation $\Delta_{mn}^{(1)}$ and type-2 triangulation $\Delta_{mn}^{(2)}$, Chui, Schumaker, and Wang [2,3] in 1982 investigated the space $S_3^{1,d}(\Delta_{mn}^{(1)})$ of cubic splines and the space $S_2^{1,d}(\Delta_{mn}^{(2)})$ of quadratic splines with boundary conditions $d = 0$ and $d = 1$, respectively. In 1985, Wang and He [4] studied the space $S_2^{1,d}(\bar{\Delta}_{mn}^{(2)})$ over a nonuniform triangulation with $d = 0, 1$. In a more general case, the dimensions and bases of the spaces $S_k^{1,0}(\Delta_{mn}^{(1)})$ with $k \geq 3$, $S_k^{1,1}(\Delta_{mn}^{(1)})$ with $k \geq 4$, and $S_k^{1,1}(\Delta_{mn}^{(2)})$ with $k \geq 3$ were given by Le in [5,6]. Based on these theoretical results, the interpolation and approximation by $S_2^{1,1}(\bar{\Delta}_{mn}^{(2)})$ and $S_3^{1,1}(\Delta_{mn}^{(2)})$ were discussed in [7–9]. As applications, the quadratic piecewise polynomials in $S_2^{1,d}(\Delta_{mn}^{(2)})$ $(d = 0, 1)$ are used as spline finite elements to solve bending problem of plates [10,11], recently. Also, over the uniform type-1 and type-2 triangulations, linear spline prewavelets are constructed and studied intensively very recently (see [12,13], and the references therein).

Similar to the univariate setting, bivariate cubic spline spaces are most useful to fit smooth data and to be not too complicated in computing In this paper, we will first construct a locally supported basis of the bivariate cubic spline space $S_3^1(\bar{\Delta}_{mn}^{(2)})$ over a nonuniform type-2 triangulation, then determine the dimension and construct a local support basis of the subspace $S_3^{1,d}(\bar{\Delta}_{mn}^{(2)})$ for $d = 0, 1$ of the spline spaces with boundary conditions.

2. BIVARIATE C^1 CUBIC SPLINE SPACE $S_3^1\left(\bar{\Delta}_{mn}^{(2)}\right)$

Approximation properties of bivariate cubic spline spaces over a uniform type-2 triangulation were studied by Lai in [14]. In this section, we construct a local basis of the space $S_3^1(\bar{\Delta}_{mn}^{(2)})$, which can be regarded as a generalization of the basis of $S_3^1(\Delta_{mn}^{(2)})$ given in [14].

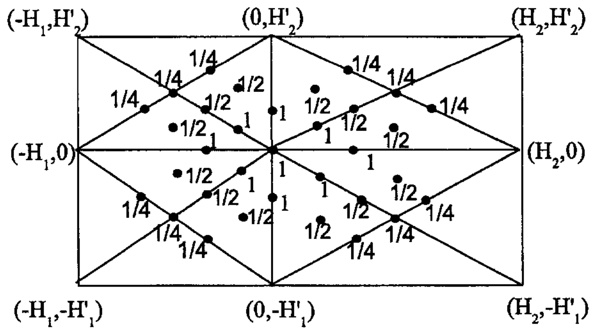


Figure 2 Nonzero B-net ordinates of $ls^{(0,0)}(x, y, H_1, H_2, H_1', H_2')$

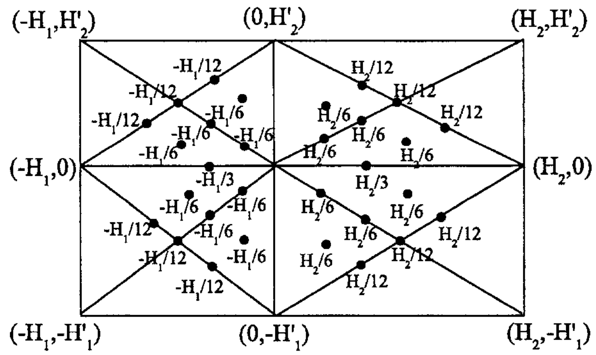


Figure 3 Nonzero B-net ordinates of $ls^{(1,0)}(x, y, H_1, H_2, H_1', H_2')$.

Let $\Gamma = \{(0,0), (1,0), (0,1)\}$ and $ls^{(i,j)}(x, y, H_1, H_2, H_1', H_2')$, $(i,j) \in \Gamma$, $ls^{(1,2)}(x, y, H_1, H_2, H_2')$, and $ls^{(2,1)}(x, y, H_1, H_1', H_2')$ the five C^1 cubic splines with their B-net representation defined on their local supports as shown in Figures 2–5, respectively, where all other B-net ordinates not shown in the figures are vanished. Then, we have the following theorem.

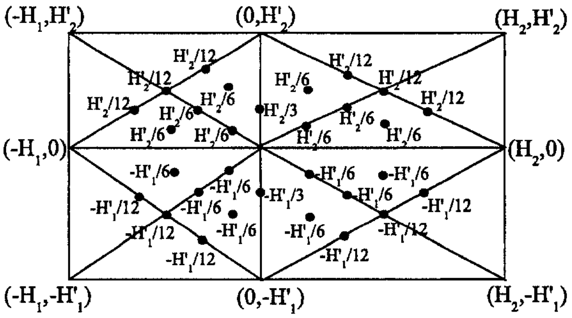
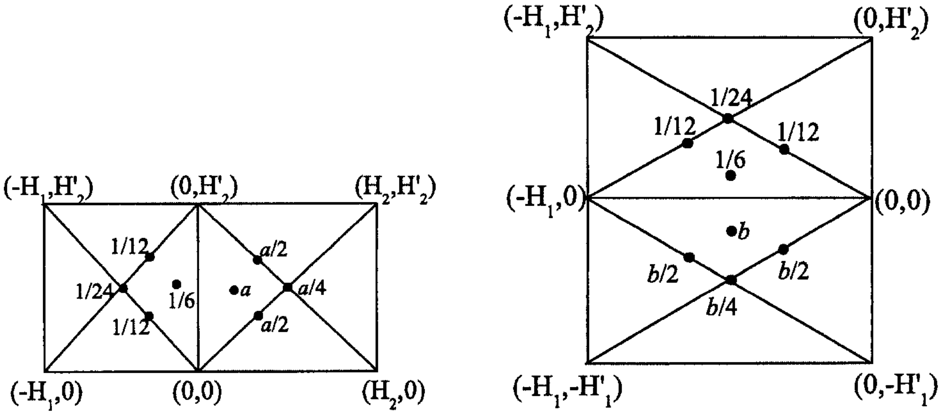


Figure 4. Nonzero B-net ordinates of $ls^{(0,1)}(x, y, H_1, H_2, H_1', H_2')$.



(a) Nonzero B-net ordinates of $ls^{(1,2)}(x, y, H_1, H_2, H_2')$ with $a = -H_2/6H_1$

(b) Nonzero B-net ordinates of $ls^{(2,1)}(x, y, H_1, H_1', H_2')$ with $b = -H_1'/6H_2'$

Figure 5.

THEOREM 1. *Given $G = [0, m] \times [0, n]$, $G_1 = [0, m] \times [0, n-1]$, $G_2 = [1, m] \times [0, n]$, and $\mathbb{Z}^2 = \{(i, j) : i$ and j are arbitrary integers$\}$. Then, the set of locally supported functions,*

$$\begin{aligned} &\left\{ls^{\alpha}\left(x - x_i, y - y_j, h_i, h_{i+1}, t_j, t_{j+1}\right) : (i, j) \in G \cap \mathbb{Z}^2, \alpha \in \Gamma\right\} \\ &\cup \left\{ls^{(1,2)}\left(x - x_i, y - y_j, h_i, h_{i+1}, t_{j+1}\right) : (i, j) \in G_1 \cap \mathbb{Z}^2\right\} \\ &\cup \left\{ls^{(2,1)}\left(x - x_i, y - y_j, h_i, t_j, t_{j+1}\right) : (i, j) \in G_2 \cap \mathbb{Z}^2\right\} \end{aligned} \tag{2}$$

becomes a basis of the space $S_3^1(\bar{\Delta}_{mn}^{(2)})$, where, without loss of generality, we set $h_0 = t_0 = h_{m+1} = t_{n+1} = 1$.

PROOF. It is well known that

$$\dim S_3^1\left(\bar{\Delta}_{mn}^{(2)}\right) = 5mn + 4m + 4n + 3. \tag{3}$$

The total number of the spline functions listed in (2) is $5mn + 4m + 4n + 3$. Therefore, it is sufficient to verify that all of them are linearly independent over Ω. For $(x, y) \in \Omega$, let

$$\begin{aligned} s(x, y) :=& \sum_{p=0}^{m} \sum_{q=0}^{n} A_{p,q} ls^{(0,0)}\left(x - x_p, y - y_q, h_p, h_{p+1}, t_q, t_{q+1}\right) \\ &+ \sum_{p=0}^{m} \sum_{q=0}^{n} B_{p,q} ls^{(1,0)}\left(x - x_p, y - y_q, h_p, h_{p+1}, t_q, t_{q+1}\right) \end{aligned} \tag{4}$$

$$+\sum_{p=0}^{m}\sum_{q=0}^{n} C_{p,q} ls^{(0,1)}\left(x-x_p, y-y_q, h_p, h_{p+1}, t_q, t_{q+1}\right)$$
$$+\sum_{p=0}^{m}\sum_{q=0}^{n-1} D_{p,q} ls^{(1,2)}\left(x-x_p, y-y_q, h_p, h_{p+1}, t_{q+1}\right) \qquad (4)(\text{cont.})$$
$$+\sum_{p=1}^{m}\sum_{q=0}^{n} E_{p,q} ls^{(2,1)}\left(x-x_p, y-y_q, h_p, t_q, t_{q+1}\right) \equiv 0.$$

Then, we have that $A_{p,q}=0$ for all $(p,q)\in G\cap\mathbb{Z}^2$ since $0=s(x_i,y_j)=A_{i,j}$, $B_{p,q}=0$ for all $(p,q)\in G\cap\mathbb{Z}^2$ since $0=\frac{\partial s}{\partial x}|_{(x_i,y_j)}=B_{i,j}$, $C_{p,q}=0$ for all $(p,q)\in G\cap\mathbb{Z}^2$ since $0=\frac{\partial s}{\partial y}|_{(x_i,y_j)}=C_{i,j}$, $D_{p,q}=0$ for all $(p,q)\in G_1\cap\mathbb{Z}^2$ since $0=\frac{\partial s}{\partial x}|_{(x_i,(y_j+y_{j+1})/2)}=-\left(t_{j+1}^2/2h_it_j^2\right)D_{i,j}$, and $E_{p,q}=0$ for all $(p,q)\in G_2\cap\mathbb{Z}^2$ since $0=\frac{\partial s}{\partial y}|_{((x_{i-1}+x_i)/2,y_j)}=(1/2t_{j+1})E_{i,j}$. This completes the proof of the theorem. ∎

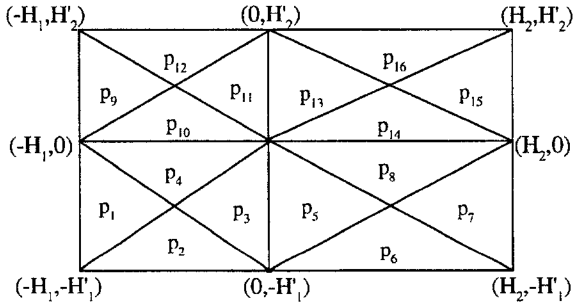


Figure 6 The support of $ls^{\alpha}(x,y,H_1,H_2,H_1',H_2')$, $\alpha\in\Gamma$

Next, for convenience in applications, we transfer the B-net representation of all the local bases in terms of Cartesian coordinates. The piecewise representation of $ls^{\alpha}(x,y,H_1,H_2,H_1',H_2')$, $\alpha\in\Gamma$, $ls^{(1,2)}(x,y,H_1,H_2,H_2')$, and $ls^{(2,1)}(x,y,H_1,H_1',H_2')$ on their supports as shown in Figures 6 and 7 can be given explicitly as follows.

(a) $ls^{(0,0)}\left(x,y,H_1,H_2,H_1',H_2'\right)$:

$$p_1^{(0,0)}=\frac{\left(H_1+x\right)^2\left(2H_1H_1'-H_1'x+3H_1y\right)}{H_1^3H_1'},$$
$$p_2^{(0,0)}=\frac{\left(H_1'+y\right)^2\left(2H_1H_1'+3H_1'x-H_1y\right)}{H_1H_1'^3},$$
$$p_3^{(0,0)}=\frac{H_1^3H_1'^3-H_1'^3x^3-3H_1H_1'^2x^2y-3H_1H_1'^3x^2-2H_1^3y^3-3H_1^3H_1'y^2}{H_1^3H_1'^3},$$
$$p_4^{(0,0)}=\frac{H_1^3H_1'^3-H_1^3y^3-3H_1^2H_1'xy^2-3H_1^3H_1'y^2-2H_1'^3x^3-3H_1H_1'^3x^2}{H_1^3H_1'^3},$$
$$p_5^{(0,0)}=\frac{H_2^3H_1'^3+H_1'^3x^3-3H_2H_1'^2x^2y-3H_2H_1'^3x^2-3H_2^3H_1'y^2-2H_2^3y^3}{H_2^3H_1'^3},$$
$$p_6^{(0,0)}=\frac{\left(H_1'+y\right)^2\left(2H_2H_1'-H_2y-3H_1'x\right)}{H_2H_1'^3},$$
$$p_7^{(0,0)}=\frac{\left(H_2-x\right)^2\left(2H_2H_1'+H_1'x+3H_2y\right)}{H_2^3H_1'},$$
$$p_8^{(0,0)}=\frac{H_2^3H_1'^3-3H_2^3H_1'y^2-3H_2H_1'^3x^2-H_2^3y^3+3H_2^2H_1'xy^2+2H_1'^3x^3}{H_2^3H_1'^3},$$

$$p_9^{(0,0)} = \frac{(H_1+x)^2(2H_1H_2' - H_2'x - 3H_1y)}{H_1^3H_2'},$$

$$p_1^{(0,0)}0 = -\frac{2H_2'^3x^3 - H_1^3H_2'^3 + 3H_1^3H_2'y^2 + 3H_1H_2'^3x^2 - H_1^3y^3 + 3H_1^2H_2'xy^2}{H_1^3H_2'^3},$$

$$p_{11}^{(0,0)} = \frac{H_1^3H_2'^3 - H_2'^3x^3 + 3H_1H_2'^2x^2y + 2H_1^3y^3 - 3H_1^3H_2'y^2 - 3H_1H_2'^3x^2}{H_1^3H_2'^3},$$

$$p_{12}^{(0,0)} = \frac{(H_2'-y)^2(2H_1H_2' + H_1y + 3H_2'x)}{H_1H_2'^3},$$

$$p_{13}^{(0,0)} = \frac{H_2^3H_2'^3 + H_2'^3x^3 + 3H_2H_2'^2x^2y + 2H_2^3y^3 - 3H_2H_2'^3x^2 - 3H_2^3H_2'y^2}{H_2^3H_2'^3},$$

$$p_{14}^{(0,0)} = \frac{H_2^3H_2'^3 + H_2^3y^3 + 3H_2^2H_2'xy^2 + 2H_2'^3x^3 - 3H_2H_2'^3x^2 - 3H_2^3H_2'y^2}{H_2^3H_2'^3},$$

$$p_{15}^{(0,0)} = \frac{(H_2-x)^2(2H_2H_2' + H_2'x - 3H_2y)}{H_2^3H_2'},$$

$$p_{16}^{(0,0)} = \frac{(H_2'-y)^2(2H_2H_2' + H_2y - 3H_2'x)}{H_2H_2'^3}.$$

(b) $ls^{(1,0)}(x, y, H_1, H_2, H_1', H_2')$:

$$p_1^{(1,0)} = -\frac{(H_1+x)^2(2H_1H_1' - H_1'x + 3H_1y)}{3H_1^2H_1'},$$

$$p_2^{(1,0)} = -\frac{(H_1'+y)^2(2H_1H_1' - H_1y + 3H_1'x)}{3H_1'^3},$$

$$p_3^{(1,0)} = \frac{3H_1^2H_1'^2 + x\left(2H_1'^2x^2 + 6H_1H_1'^2x + 3H_1H_1'xy - 3H_1^2y^2\right)}{3H_1^2H_1'^2},$$

$$p_4^{(1,0)} = \frac{-H_1^3y^3 + 3H_1^2H_1'^3x + 6H_1H_1'^3x^2 + 3H_1'^3x^3}{3H_1^2H_1'^3},$$

$$p_5^{(1,0)} = \frac{x\left(3H_2^2H_1'^2 + 2H_1'^2x^2 - 3H_2H_1'xy - 6H_2H_1'^2x - 3H_2^2y^2\right)}{3H_2^2H_1'^2},$$

$$p_6^{(1,0)} = \frac{(H_1'+y)^2(2H_2H_1' - H_2y - 3H_1'x)}{3H_1'^3},$$

$$p_7^{(1,0)} = \frac{(H_2-x)^2(2H_2H_1' + H_1'x + 3H_2y)}{3H_2^2H_1'},$$

$$p_8^{(1,0)} = \frac{H_2^3y^3 + 3H_2^2H_1'^3x - 6H_2H_1'^3x^2 + 3H_1'^3x^3}{3H_2^2H_1'^3},$$

$$p_9^{(1,0)} = -\frac{(H_1+x)^2(2H_1H_2' - H_2'x - 3H_1y)}{3H_1^2H_2'},$$

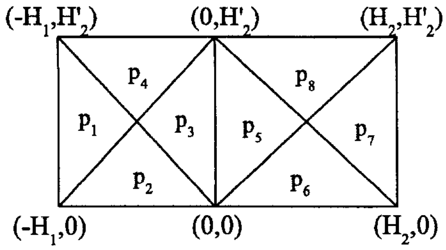
$$p_{10}^{(1,0)} = \frac{H_1^3y^3 + 3H_1^2H_2'^3x + 6H_1H_2'^3x^2 + 3H_2'^3x^3}{3H_1^2H_2'^3},$$

$$p_{11}^{(1,0)} = \frac{x\left(3H_1^2H_2'^2 + 2H_2'^2x^2 - 3H_1H_2'xy + 6H_1H_2'^2x - 3H_1^2y^2\right)}{3H_1^2H_2'^2},$$

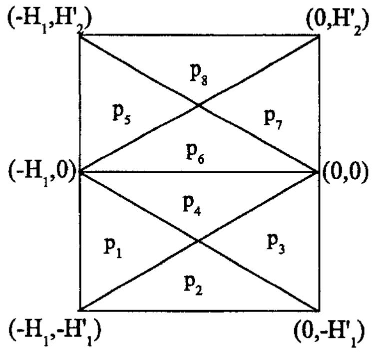
$$p_{12}^{(1,0)} = -\frac{(H_2'-y)^2(2H_1H_2' + H_1y + 3H_2'x)\,3}{H_2'^3},$$

$$p_{13}^{(1,0)} = -\frac{x\left(-3H_2^2H_2'^2 - 2H_2'^2x^2 + 6H_2H_2'^2x - 3H_2H_2'xy + 3H_2^2y^2\right)}{3H_2^2H_2'^2},$$

$$p_{14}^{(1,0)} = -\frac{H_2^3y^3 - 3H_2^2H_2'^3x + 6H_2H_2'^3x^2 - 3H_2'^3x^3}{3H_2^2H_2'^3},$$



(a) The support of $ls^{(1,2)}(x, y, H_1, H_2, H_2')$



(b) The support of $ls^{(2,1)}(x, y, H_1, H_1', H_2')$

Figure 7

$$p_{15}^{(1,0)} = \frac{(-H_2+x)^2\,(2H_2H_2' + H_2'x - 3H_2y)}{3H_2^2H_2'},$$

$$p_{16}^{(1,0)} = -\frac{(H_2'-y)^2\,(-2H_2H_2' - H_2y + 3H_2'x)}{3H_2'^3}.$$

(c) $ls^{(0,1)}(x, y, H_1, H_2, H_1', H_2')$:

$$p_1^{(0,1)} = \frac{(H_1+x)^2\,(-2H_1H_1' + H_1'x - 3H_1y)}{3H_1^3},$$

$$p_2^{(0,1)} = \frac{(H_1'+y)^2\,(-2H_1H_1' + H_1y - 3H_1'x)}{3H_1H_1'^2},$$

$$p_3^{(0,1)} = \frac{-H_1'^3x^3 + 3H_1^3H_1'^2y + 6H_1^3H_1'y^2 + 3H_1^3y^3}{3H_1^3H_1'^2},$$

$$p_4^{(0,1)} = -\frac{y\left(-3H_1^2H_1'^2 - 2H_1^2y^2 - 3H_1H_1'xy - 6H_1^2H_1'y + 3H_1'^2x^2\right)}{3H_1^2H_1'^2},$$

$$p_5^{(0,1)} = \frac{H_1'^3x^3 + 3H_2^3H_1'^2y + 6H_2^3H_1'y^2 + 3H_2^3y^3}{3H_2^3H_1'^2},$$

$$p_6^{(0,1)} = -\frac{(H_1'+y)^2\,(2H_2H_1' - H_2y - 3H_1'x)}{3H_2H_1'^2},$$

$$p_7^{(0,1)} = -\frac{(H_2-x)^2\,(2H_2H_1' + H_1'x + 3H_2y)}{3H_2^3},$$

$$p_8^{(0,1)} = \frac{y\left(3H_2^2H_1'^2 + 2H_2^2y^2 + 6H_2^2H_1'y - 3H_2H_1'xy - 3H_1'^2x^2\right)}{3H_2^2H_1'^2},$$

$$p_9^{(0,1)} = \frac{(H_1+x)^2\,(2H_1H_2' - H_2'x - 3H_1y)}{3H_1^3},$$

$$p_{10}^{(0,1)} = \frac{y\left(2H_1^2y^2 - 6H_1^2H_2'y - 3H_1H_2'xy + 3H_1^2H_2'^2 - 3H_2'^2x^2\right)}{3H_1^2H_2'^2},$$

$$p_{11}^{(0,1)} = \frac{H_2'^3x^3 + 3H_1^3H_2'^2y - 6H_1^3H_2'y^2 + 3H_1^3y^3}{3H_1^3H_2'^2},$$

$$p_{12}^{(0,1)} = \frac{(-H_2'+y)^2\,(2H_1H_2' + H_1y + 3H_2'x)}{3H_1H_2'^2},$$

$$p_{13}^{(0,1)} = \frac{-H_2'^3x^3 + 3H_2^3H_2'^2y - 6H_2^3H_2'y^2 + 3H_2^3y^3}{3H_2^3H_2'^2},$$

$$p_{14}^{(0,1)} = \frac{y\left(3H_2^2H_2'^2 + 2H_2^2y^2 + 3H_2H_2'xy - 6H_2^2H_2'y - 3H_2'^2x^2\right)}{3H_2^2H_2'^2},$$

$$p_{15}^{(0,1)} = \frac{(-H_2+x)^2\left(2H_2H_2' + H_2'x - 3H_2y\right)}{3H_2^3},$$

$$p_{16}^{(0,1)} = \frac{(H_2'-y)^2\left(2H_2H_2' + H_2y - 3H_2'x\right)}{3H_2H_2'^2},$$

(d) $ls^{(1,2)}(x, y, H_1, H_2, H_2')$:

$$p_1^{(1,2)} = \frac{(H_1+x)^3}{3H_1^3},$$

$$p_2^{(1,2)} = \frac{y^2\left(-2H_1y + 3H_2'H_1 + 3H_2'x\right)}{3H_1H_2'^3},$$

$$p_3^{(1,2)} = -\frac{x\left(H_2'^2x^2 + 3H_2'^2H_1x + 6H_1^2H_2'y - 6H_1^2y^2\right)}{3H_1^3H_2'^2},$$

$$p_4^{(1,2)} = \frac{(H_2'-y)^2\left(H_2'H_1 + 2H_1y + 3H_2'x\right)}{3H_1H_2'^3},$$

$$p_5^{(1,2)} = \frac{x\left(-H_2'^2x^2 + 3H_2'^2H_2x - 6H_2^2H_2'y + 6H_2^2y^2\right)}{3H_1H_2^2H_2'^2},$$

$$p_6^{(1,2)} = -\frac{y^2\left(-2H_2y + 3H_2H_2' - 3H_2'x\right)}{3H_1H_2'^3},$$

$$p_7^{(1,2)} = -\frac{(H_2-x)^3}{3H_1H_2^2},$$

$$p_8^{(1,2)} = -\frac{(-H_2'+y)^2\left(H_2H_2' + 2H_2y - 3H_2'x\right)}{3H_1H_2'^3}.$$

(e) $ls^{(2,1)}(x, y, H_1, H_1', H_2')$:

$$p_1^{(2,1)} = -\frac{(H_1+x)^2\left(H_1'H_1 - 2H_1'x + 3H_1y\right)}{3H_1^3H_2'},$$

$$p_2^{(2,1)} = -\frac{(H_1'+y)^3}{3H_1'^2H_2'},$$

$$p_3^{(2,1)} = -\frac{x^2\left(2H_1'x + 3H_1'H_1 + 3H_1y\right)}{3H_1^3H_2'},$$

$$p_4^{(2,1)} = \frac{y\left(H_1^2y^2 + 3H_1^2H_1'y - 6H_1'^2H_1x - 6H_1'^2x^2\right)}{3H_1^2H_1'^2H_2'},$$

$$p_5^{(2,1)} = -\frac{(H_1+x)^2\left(-H_1H_2' + 2H_2'x + 3H_1y\right)}{3H_1^3H_2'},$$

$$p_6^{(2,1)} = \frac{y\left(H_1^2y^2 - 3H_1^2H_2'y - 6H_2'^2H_1x - 6H_2'^2x^2\right)}{3H_1^2H_2'^3},$$

$$p_7^{(2,1)} = \frac{x^2\left(2H_2'x + 3H_1H_2' - 3H_1y\right)}{3H_1^3H_2'},$$

$$p_8^{(2,1)} = \frac{(H_2'-y)^3}{3H_2'^3}.$$

3. THE SPACE $S_3^{1,0}\left(\bar{\triangle}_{mn}^{(2)}\right)$

It can be easily seen from (1) that

$$S_3^{1,0}\left(\bar{\triangle}_{mn}^{(2)}\right) = \left\{s \in S_3^1\left(\bar{\triangle}_{mn}^{(2)}\right) : s\Big|_{\partial\Omega} \equiv 0\right\}. \qquad (5)$$

For any $s(x,y) \in S_3^{1,0}(\bar{\Delta}_{mn}^{(2)}) \subset S_3^1(\bar{\Delta}_{mn}^{(2)})$, we have from Theorem 1 that

$$\begin{aligned} s(x,y) = &\sum_{p=0}^{m}\sum_{q=0}^{n} A_{p,q} ls^{(0,0)}(x-x_p, y-y_q, h_p, h_{p+1}, t_q, t_{q+1}) \\ &+\sum_{p=0}^{m}\sum_{q=0}^{n} B_{p,q} ls^{(1,0)}(x-x_p, y-y_q, h_p, h_{p+1}, t_q, t_{q+1}) \\ &+\sum_{p=0}^{m}\sum_{q=0}^{n} C_{p,q} ls^{(0,1)}(x-x_p, y-y_q, h_p, h_{p+1}, t_q, t_{q+1}) \\ &+\sum_{p=0}^{m}\sum_{q=0}^{n-1} D_{p,q} ls^{(1,2)}(x-x_p, y-y_q, h_p, h_{p+1}, t_{q+1}) \\ &+\sum_{p=1}^{m}\sum_{q=0}^{n} E_{p,q} ls^{(2,1)}(x-x_p, y-y_q, h_p, t_q, t_{q+1}). \end{aligned} \tag{6}$$

Note that, for $x_{i-1} \le x \le x_i$, $i = 1, \dots, m$, $y = 0$, there only exist seven nonzero spline functions among all $5mn + 4m + 4n + 3$ basis functions in (6), we have

$$\begin{aligned} s(x,y) = &A_{i-1,0} p_{14}^{(0,0)}(x-x_{i-1}, 0, h_{i-1}, h_i, t_0, t_1) + A_{i,0} p_{10}^{(0,0)}(x-x_i, 0, h_i, h_{i+1}, t_0, t_1) \\ &+ B_{i-1,0} p_{14}^{(1,0)}(x-x_{i-1}, 0, h_{i-1}, h_i, t_0, t_1) + B_{i,0} p_{10}^{(1,0)}(x-x_i, 0, h_i, h_{i+1}, t_0, t_1) \\ &+ C_{i-1,0} p_{14}^{(0,1)}(x-x_{i-1}, 0, h_{i-1}, h_i, t_0, t_1) + C_{i,0} p_{10}^{(0,1)}(x-x_i, 0, h_i, h_{i+1}, t_0, t_1) \\ &+ E_{i,0} p_6^{(2,1)}(x-x_i, 0, h_i, t_0, t_1) \\ = &A_{i-1,0} p_{14}^{(0,0)}(x-x_{i-1}, 0, h_{i-1}, h_i, t_0, t_1) + A_{i,0} p_{10}^{(0,0)}(x-x_i, 0, h_i, h_{i+1}, t_0, t_1) \\ &+ B_{i-1,0} p_{14}^{(1,0)}(x-x_{i-1}, 0, h_{i-1}, h_i, t_0, t_1) + B_{i,0} p_{10}^{(1,0)}(x-x_i, 0, h_i, h_{i+1}, t_0, t_1). \end{aligned}$$

It yields from (5) that

$$s(x,y)\big|_{x_{i-1}\le x\le x_i,\ y=0} \equiv 0, \qquad i = 1, \dots, m,$$

that is, for $i = 1, \dots, m$,

$$\begin{aligned} &A_{i-1,0}(x-x_i)^2(x_i - 3x_{i-1} + 2x) - A_{i,0}(x-x_{i-1})^2(2x - 3x_i + x_{i-1}) \\ &+B_{i-1,0}(x-x_{i-1})(x-x_i)^2 h_i + B_{i,0}(x-x_i)(x-x_{i-1})^2 h_i \equiv 0. \end{aligned} \tag{7}$$

By substituting $x = x_{i-1}$, $x = x_i$, $x = x_{i-1} + (1/3)(x_i - x_{i-1})$, and $x = x_{i-1} + (2/3)(x_i - x_{i-1})$ into (7), respectively, we can obtain for $i = 1, \dots, m$,

$$\begin{aligned} A_{i-1,0} = A_{i,0} &= 0, \\ 4h_i B_{i-1,0} - 2h_i B_{i,0} + 7A_{i,0} + 20A_{i-1,0} &= 0, \\ 2h_i B_{i-1,0} - 4h_i B_{i,0} + 20A_{i,0} + 7A_{i-1,0} &= 0. \end{aligned} \tag{8}$$

Therefore,

$$A_{i,0} = B_{i,0} = 0, \qquad i = 0, 1, \dots, m. \tag{9}$$

Similarly, we have

$$s(x,y)\big|_{x_{i-1}\le x\le x_i,\ y=y_n} \equiv 0, \qquad i = 1, \dots, m,$$

and thus,

$$A_{i,n} = 0, \qquad B_{i,n} = 0, \quad i = 0, 1, \dots, m. \tag{10}$$

We also obtain

$$s(x,y)\,|_{x=0,\ y_{j-1}\le y\le y_j} \equiv 0, \qquad j=1,\dots,n,$$

hence,

$$A_{0,j}=0, \qquad C_{0,j}=0, \quad j=0,1,\dots,n. \tag{11}$$

And

$$s(x,y)\,|_{x=x_m,\ y_{j-1}\le y\le y_j} \equiv 0, \qquad j=1,\dots,n,$$

thus,

$$A_{m,j}=0, \qquad C_{m,j}=0, \quad j=0,1,\dots,n. \tag{12}$$

Finally, we obtain

$$\begin{aligned} &A_{i,0}=A_{i,n}=0, && i=0,1,\dots,m,\\ &A_{0,j}=A_{m,j}=0, && j=1,\dots,n-1,\\ &B_{i,0}=B_{i,n}=0, && i=0,1,\dots,m,\\ &C_{0,j}=C_{m,j}=0, && j=0,1,\dots,n. \end{aligned} \tag{13}$$

Therefore, there are $4m+4n+4$ parameters determined by the boundary conditions. This means that

$$\dim S_3^{1,0}\left(\bar{\Delta}_{mn}^{(2)}\right) \le \dim S_3^1\left(\bar{\Delta}_{mn}^{(2)}\right) - (4m+4n+4) = 5mn-1. \tag{14}$$

Noticing the linear independence of $5mn-1$ basis functions given below, we get the following result.

THEOREM 2.

$$\dim S_3^{1,0}\left(\bar{\Delta}_{mn}^{(2)}\right) = 5mn-1, \tag{15}$$

and the set

$$\begin{aligned} &\left\{ls^{(0,0)}\left(x-x_i,y-y_j,h_i,h_{i+1},t_j,t_{j+1}\right):\ i=1,\dots,m-1,\ j=1,\dots,n-1\right\}\\ &\cup\left\{ls^{(1,0)}\left(x-x_i,y-y_j,h_i,h_{i+1},t_j,t_{j+1}\right):\ i=0,1,\dots,m,\ j=1,\dots,n-1\right\}\\ &\cup\left\{ls^{(0,1)}\left(x-x_i,y-y_j,h_i,h_{i+1},t_j,t_{j+1}\right):\ i=1,\dots,m-1,\ j=0,1,\dots,n\right\}\\ &\cup\left\{ls^{(1,2)}\left(x-x_i,y-y_j,h_i,h_{i+1},t_{j+1}\right):\ i=0,1\dots,m,\ j=0,1,\dots,n-1\right\}\\ &\cup\left\{ls^{(2,1)}\left(x-x_i,y-y_j,h_i,t_j,t_{j+1}\right):\ i=1,\dots,m,\ j=0,1,\dots,n\right\}, \end{aligned} \tag{16}$$

forms a local basis of $S_3^{1,0}\left(\bar{\Delta}_{mn}^{(2)}\right)$.

4. THE SPACE $S_3^{1,1}\left(\bar{\Delta}_{mn}^{(2)}\right)$

From the definition given in (1), we have

$$\begin{aligned} S_3^{1,1}\left(\bar{\Delta}_{mn}^{(2)}\right) &= \left\{s\in S_3^{1,0}\left(\bar{\Delta}_{mn}^{(2)}\right):\ \left.\frac{\partial s}{\partial \mathbf{n}}\right|_{\partial\Omega}\equiv 0\right\}\\ &= \left\{s\in S_3^{1,0}\left(\bar{\Delta}_{mn}^{(2)}\right):\ s_x(0,y)=s_x(x_m,y)\equiv 0,\ s_y(x,0)=s_y(x,y_n)\equiv 0\right\}. \end{aligned} \tag{17}$$

From Theorem 2 in Section 3, we have that $s(x,y) \in S_3^{1,0}(\bar{\Delta}_{mn}^{(2)})$ if and only if $s(x,y)$ can be expressed as the following,

$$
\begin{aligned}
s(x,y) = & \sum_{p=1}^{m-1}\sum_{q=1}^{n-1} A_{p,q} ls^{(0,0)}(x-x_p, y-y_q, h_p, h_{p+1}, t_q, t_{q+1}) \\
& + \sum_{p=0}^{m}\sum_{q=1}^{n-1} B_{p,q} ls^{(1,0)}(x-x_p, y-y_q, h_p, h_{p+1}, t_q, t_{q+1}) \\
& + \sum_{p=1}^{m-1}\sum_{q=0}^{n} C_{p,q} ls^{(0,1)}(x-x_p, y-y_q, h_p, h_{p+1}, t_q, t_{q+1}) \\
& + \sum_{p=0}^{m}\sum_{q=0}^{n-1} D_{p,q} ls^{(1,2)}(x-x_p, y-y_q, h_p, h_{p+1}, t_{q+1}) \\
& + \sum_{p=1}^{m}\sum_{q=0}^{n} E_{p,q} ls^{(2,1)}(x-x_p, y-y_q, h_p, t_q, t_{q+1}).
\end{aligned} \tag{18}
$$

When $s(x,y)$ is restricted on the triangular element T: $x_{i-1} \le x \le x_i$, $t_1 x + h_i y \le t_1 x_i$, and $t_1 x - h_i y \ge t_1 x_{i-1}$, we further have

$$
\begin{aligned}
s(x,y)|_T = & C_{i-1,0} p_{14}^{(0,1)}(x-x_{i-1}, y, h_{i-1}, h_i, t_0, t_1) + C_{i,0} p_{10}^{(0,1)}(x-x_i, y, h_i, h_{i+1}, t_0, t_1) \\
& + A_{i-1,1} p_6^{(0,0)}(x-x_{i-1}, y-y_1, h_{i-1}, h_i, t_1, t_2) \\
& + A_{i,1} p_2^{(0,0)}(x-x_i, y-y_1, h_i, h_{i+1}, t_1, t_2) \\
& + B_{i-1,1} p_6^{(1,0)}(x-x_{i-1}, y-y_1, h_{i-1}, h_i, t_1, t_2) \\
& + B_{i,1} p_2^{(1,0)}(x-x_i, y-y_1, h_i, h_{i+1}, t_1, t_2) \\
& + C_{i-1,1} p_6^{(0,1)}(x-x_{i-1}, y-y_1, h_{i-1}, h_i, t_1, t_2) \\
& + C_{i,1} p_2^{(0,1)}(x-x_i, y-y_1, h_i, h_{i+1}, t_1, t_2) \\
& + D_{i-1,0} p_6^{(1,2)}(x-x_{i-1}, y, h_{i-1}, h_i, t_1) + D_{i,0} p_2^{(1,2)}(x-x_i, y, h_i, h_{i+1}, t_1) \\
& + E_{i,0} p_6^{(2,1)}(x-x_i, y, h_i, t_0, t_1) + E_{i,1} p_2^{(2,1)}(x-x_i, y-y_1, h_i, t_1, t_2) \\
= & C_{i-1,0} \frac{y\left[3h_i^2 t_1^2 + 2h_i^2 y^2 + 3h_i t_1 (x-x_{i-1}) y - 6h_i^2 t_1 y - 3t_1^2 (x-x_{i-1})^2\right]}{3h_i^2 t_1^2} \\
& + C_{i,0} \frac{y\left[2h_i^2 y^2 - 6h_i^2 t_1 y - 3h_i t_1 (x-x_i) y + 3h_i^2 t_1^2 - 3t_1^2 (x-x_i)^2\right]}{3h_i^2 t_1^2} \\
& + A_{i-1,1} \frac{y^2\left[2h_i t_1 - h_i (y-y_1) - 3t_1 (x-x_{i-1})\right]}{h_i t_1^3} \\
& + A_{i,1} \frac{y^2\left[2h_i t_1 + 3t_1 (x-x_i) - h_i (y-y_1)\right]}{h_i t_1^3} \\
& + B_{i-1,1} \frac{y^2\left[2h_i t_1 - h_i (y-y_1) - 3t_1 (x-x_{i-1})\right]}{3t_1^3} \\
& - B_{i,1} \frac{y^2\left[2h_i t_1 - h_i (y-y_1) + 3t_1 (x-x_i)\right]}{3t_1^3} \\
& - C_{i-1,1} \frac{y^2\left[2h_i t_1 - h_i (y-y_1) - 3t_1 (x-x_{i-1})\right]}{3h_i t_1^2} \\
& + C_{i,1} \frac{y^2\left[-2h_i t_1 + h_i (y-y_1) - 3t_1 (x-x_i)\right]}{3h_i t_1^2}
\end{aligned}
$$

$$- D_{i-1,0}\frac{y^2\left[-2h_iy+3h_it_1-3t_1\left(x-x_{i-1}\right)\right]}{3h_{i-1}t_1^3}$$
$$+ D_{i,0}\frac{y^2\left[-2h_iy+3t_1h_i+3t_1\left(x-x_i\right)\right]}{3h_it_1^3}$$
$$+ E_{i,0}\frac{y\left[h_i^2y^2-3h_i^2t_1y-6t_1^2h_i\left(x-x_i\right)-6t_1^2\left(x-x_i\right)^2\right]}{3h_i^2t_1^3} - E_{i1}\frac{y^3}{3t_1^2t_2}.$$

Hence, for $i=1,\dots,m$, we have

$$\begin{aligned} s_y\left(x,y\right)|_{x_{i-1}\le x\le x_i, y=0} &= -\frac{1}{h_i^2}C_{i-1,0}\left[h_i^2-\left(x-x_{i-1}\right)^2\right] \\ &\quad + \frac{1}{h_i^2}C_{i,0}\left[h_i^2-\left(x-x_i\right)^2\right] \\ &\quad - \frac{2}{h_i^2t_1}E_{i,0}\left[h_i\left(x-x_i\right)+\left(x-x_i\right)^2\right] \\ &= \frac{1}{h_i^2}\left[C_{i-1,0}\left(x-x_i\right)\left(x-x_{i-1}+h_i\right)\right. \\ &\quad \left.+C_{i,0}\left(x-x_{i-1}\right)\left(x-x_i-h_i\right)\right] \\ &\quad + \frac{1}{h_i^2t_1}\left[2E_{i,0}\left(x-x_i\right)\left(x-x_{i-1}\right)\right]. \end{aligned} \tag{20}$$

Combining with (17), it follows that

$$C_{i-1,0}\left(x-x_i\right)\left(x-x_{i-1}+h_i\right)t_1+C_{i,0}\left(x-x_{i-1}\right)\left(x-x_i-h_i\right)t_1+2E_{i,0}\left(x-x_i\right)\left(x-x_{i-1}\right)\equiv 0. \tag{21}$$

By substituting $x=x_{i-1}$ and $x=x_{i-1}+h_i/2$ into (21), respectively, we obtain for $i=1,\dots,m$,

$$C_{i-1,0}=0, \qquad 3t_1C_{i-1,0}+3t_1C_{i,0}+2E_{i,0}=0. \tag{22}$$

It should be noted that $C_{0,0}=C_{m,0}=0$ in (13). Thus, (22) is equivalent to

$$\begin{aligned} C_{i,0}&=0, \qquad i=1,\dots,m-1, \\ E_{i,0}&=0, \qquad i=1,\dots,m. \end{aligned} \tag{23}$$

Similarly, we have

$$s_y\left(x,y\right)\big|_{x_{i-1}\le x\le x_i,\ y=y_n}\equiv 0, \qquad i=1,\dots,m,$$

and thus,

$$\begin{aligned} C_{i,n}&=0, \qquad i=1,\dots,m-1, \\ E_{i,n}&=0, \qquad i=1,\dots,m. \end{aligned} \tag{24}$$

We obtain

$$s_x\left(x,y\right)\big|_{x=0,\ y_{j-1}\le y\le y_j}\equiv 0, \qquad j=1,\dots,n,$$

hence,

$$\begin{aligned} B_{0,j}&=0, \qquad j=1,\dots,n-1, \\ D_{0,j}&=0, \qquad j=0,1,\dots,n-1. \end{aligned} \tag{25}$$

Also,

$$s_x\left(x,y\right)\big|_{x=x_m,\ y_{j-1}\le y\le y_j}\equiv 0, \qquad j=1,\dots,n,$$

and thus,

$$\begin{aligned} B_{m,j}&=0, \qquad j=1,\dots,n-1, \\ D_{m,j}&=0, \qquad j=0,1,\dots,n-1. \end{aligned} \tag{26}$$

Finally, we obtain

$$\begin{aligned} B_{0,j} &= 0, & B_{m,j} &= 0, & j &= 1, \dots, n-1, \\ C_{i,0} &= 0, & C_{i,n} &= 0, & i &= 1, \dots, m-1, \\ D_{0,j} &= 0, & D_{m,j} &= 0, & j &= 0, 1, \dots, n-1, \\ E_{i,0} &= 0, & E_{i,n} &= 0, & i &= 1, \dots, m. \end{aligned} \tag{27}$$

Therefore, there are $4m+4n-4$ coefficients determined by the boundary conditions. This means that, for any $s(x,y) \in S_3^{1,1}(\bar{\Delta}_{mn}^{(2)})$, $s(x,y)$ can be expressed as a linear combination of the following $5mn - 4m - 4n + 3$ functions,

$$\begin{aligned} & ls^{(0,0)}\left(x - x_i, y - y_j, h_i, h_{i+1}, t_j, t_{j+1}\right), \\ & ls^{(1,0)}\left(x - x_i, y - y_j, h_i, h_{i+1}, t_j, t_{j+1}\right), \\ & ls^{(0,1)}\left(x - x_i, y - y_j, h_i, h_{i+1}, t_j, t_{j+1}\right), \quad i = 1, \dots, m-1, \quad j = 1, \dots, n-1, \\ & ls^{(1,2)}\left(x - x_i, y - y_j, h_i, h_{i+1}, t_{j+1}\right), \quad i = 1, \dots, m-1, \quad j = 0, 1, \dots, n-1, \\ & ls^{(2,1)}\left(x - x_i, y - y_j, h_i, t_j, t_{j+1}\right), \quad i = 1, \dots, m, \quad j = 1, \dots, n-1. \end{aligned} \tag{28}$$

In other words,

$$\dim S_3^{1,1}\left(\bar{\Delta}_{mn}^{(2)}\right) \le \dim S_3^{1,0}\left(\bar{\Delta}_{mn}^{(2)}\right) - (4m + 4n - 4) = 5mn - 4m - 4n + 3. \tag{29}$$

It is clear that all of the functions listed in (28) are linearly independent, thus, we have obtained the following theorem.

THEOREM 3.

$$\dim S_3^{1,1}\left(\bar{\Delta}_{mn}^{(2)}\right) = 5mn - 4m - 4n + 3,$$

and the set of functions listed in (28) forms a basis of $S_3^{1,1}(\bar{\Delta}_{mn}^{(2)})$.

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