

Chapter 2 Determinants

2.1 Determinants by Cofactor Expansion

• If A is a square matrix, then M_{ij} (the **minor** of entry a_{ij}) is defined to be the determinant of the submatrix that remains after the i th row and j th column are deleted from A . Furthermore, C_{ij} (the **cofactor** of entry a_{ij}) is given by $C_{ij} = (-1)^{i+j}M_{ij}$.

Example. If $A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & -1 & 2 \\ -3 & 1 & 5 \end{bmatrix}$, then $M_{12} = 5(4) - (-3)(2) = 26$ and $C_{12} = -26$.

- For $n \times n$ matrix A , $\det(A) = \sum_{j=1}^n a_{ij}C_{ij}$ for any i from 1 through n
(cofactor expansion along the i th row)

or

$$\det(A) = \sum_{i=1}^n a_{ij}C_{ij} \text{ for any } j \text{ from 1 through } n$$

(cofactor expansion along the j th column)

Example. For A in the previous example, and using cofactor expansion along the first row, we obtain

$$\begin{aligned} \det(A) &= 2C_{11} + 1C_{12} + 0C_{13} \\ &= 2(-7) + 1(-26) + 0 \\ &= -40 \end{aligned}$$

- For $n \times n$ matrix A , the **adjoint** of A is given by

$$\text{adj}(A) = \text{transpose} \left(\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \right)$$

Example. For $A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & -1 & 2 \\ -3 & 1 & 5 \end{bmatrix}$,

$$\begin{aligned} \text{adj}(A) &= \text{transpose} \left(\begin{bmatrix} -7 & -26 & 1 \\ -5 & 10 & -5 \\ 2 & -4 & -6 \end{bmatrix} \right) \\ &= \begin{bmatrix} -7 & -5 & 2 \\ -26 & 10 & -4 \\ 1 & -5 & -6 \end{bmatrix} \end{aligned}$$

- If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A).$$

Example. For A in previous example, $A^{-1} = -\frac{1}{40} \begin{bmatrix} -7 & -5 & 2 \\ -26 & 10 & -4 \\ 1 & -5 & -6 \end{bmatrix}$.

- The determinant of a triangular matrix is the product of the main diagonal entries.

Example. If $A = \begin{bmatrix} -7 & 0 & 0 \\ -26 & 10 & 0 \\ 1 & -5 & -6 \end{bmatrix}$, $\det(A) = (-7)(10)(-6) = 420$.

• **Cramer's Rule.** If $A\mathbf{x} = \mathbf{b}$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution given by

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \dots, \quad x_n = \frac{\det(A_n)}{\det(A)},$$

where A_j is the matrix obtained by replacing the entries in the j th column of A by the entries in column vector \mathbf{b} .

Example. Use Cramer's rule to solve the system

$$\begin{aligned} 3x - 2y &= 12 \\ -x + 4y &= 8. \end{aligned}$$

which is $\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$ in matrix form.

Here, $A = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$, $A_1 = \begin{bmatrix} 12 & -2 \\ 8 & 4 \end{bmatrix}$, and $A_2 = \begin{bmatrix} 3 & 12 \\ -1 & 8 \end{bmatrix}$.

Finding determinants, we obtain $\det(A) = 10$, $\det(A_1) = 64$, and $\det(A_2) = 36$ and thus

$$x = \frac{64}{10} \text{ and } y = \frac{36}{10}.$$

2.2 Evaluating Determinants by Row Reduction

- If A has a row (or column) of zeros, then $\det(A) = 0$.

Example. If $A = \begin{bmatrix} 2 & 0 & -1 \\ -5 & 0 & -3 \\ 1 & 0 & 2 \end{bmatrix}$, the $\det(A) = 0$.

- $\det(A) = \det(A^T)$
(since cofactor expansion along the first row of A is the same as cofactor expansion along the first column of A^T)

- If A is a triangular matrix, then $\det(A) = a_{11} a_{22} \dots a_{nn}$.

- $\det \begin{pmatrix} A_1 \\ kA_2 \\ A_3 \end{pmatrix} = k \det \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$, etc.

Example. $\det \begin{bmatrix} 2 & 2 & 4 \\ -5 & 1 & -3 \\ 1 & 0 & 2 \end{bmatrix} = 2 \det \begin{bmatrix} 1 & 1 & 2 \\ -5 & 1 & -3 \\ 1 & 0 & 2 \end{bmatrix}$,

- $\det \begin{pmatrix} A_2 \\ A_1 \\ A_3 \end{pmatrix} = -\det \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$, etc.

Example. $\det \begin{bmatrix} 1 & 1 & 2 \\ -5 & 1 & -3 \\ 1 & 0 & 2 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & 2 \\ -5 & 1 & -3 \\ 1 & 1 & 2 \end{bmatrix}$

- $\det \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \det \begin{pmatrix} A_1 \\ kA_1 + A_2 \\ A_3 \end{pmatrix}$, etc.

Example. $\det \begin{bmatrix} 1 & 0 & 2 \\ -5 & 1 & -3 \\ 1 & 1 & 2 \end{bmatrix} = \det \begin{bmatrix} 0 & -1 & 0 \\ -5 & 1 & -3 \\ 1 & 1 & 2 \end{bmatrix}$

- If A has two proportional rows (or columns), then $\det(A) = 0$.

$$\begin{vmatrix} 2 & 14 \\ 1 & 7 \end{vmatrix} = 0$$

2.3 Properties of the Determinant Function

- $\det(kA) = k^n \det(A)$
- $\det(AB) = \det(A)\det(B)$
- A square matrix A is invertible iff $\det(A) \neq 0$.
- $\det(A^{-1}) = \frac{1}{\det(A)}$
- linear system of the form : $A\mathbf{x} = \lambda\mathbf{x} \Leftrightarrow (\lambda\mathbf{I} - A)\mathbf{x} = \mathbf{0}$
which has a nontrivial solution iff $\det(\lambda\mathbf{I} - A) = 0$ (the **characteristic equation** of A)
values for λ are called **eigenvalues** of A and
nontrivial solutions to system are called the **eigenvectors** of A corresponding to λ
- Theorem. For $n \times n$ A , the following are equivalent:
 - (a) A is invertible
 - (b) $A\mathbf{x} = \mathbf{0}$
 - (c) $\text{rref}(A) = I_n$
 - (d) A is expressible as a product of elementary matrices
 - (e) $A\mathbf{x} = \mathbf{b}$ is consistent for every vector \mathbf{b}
 - (f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector \mathbf{b}
 - (h) $\det(A) \neq 0$

2.4 A Combinatorial Approach to Determinants

- permutations, example (1, 6, 3, 2, 5, 4)
- number of inversions in a permutation
- even/odd permutations
- elementary product: $a_{1j_1} \cdot a_{2j_2} \cdot \dots \cdot a_{nj_n}$
- $n \times n$ matrix has $n!$ elementary products
- signed elementary product $\pm a_{1j_1} \cdot a_{2j_2} \cdot \dots \cdot a_{nj_n}$
use "+" if permutation (j_1, j_2, \dots, j_n) is even
use "-" if permutation (j_1, j_2, \dots, j_n) is odd
- For a square matrix A ,

$$\det(A) = \sum (\text{all signed elementary products from } A)$$

$$= \sum \pm a_{1j_1} \cdot a_{2j_2} \cdot \dots \cdot a_{nj_n}$$
- $|A| = \det(A)$
- $\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ may be obtained by cross – diagonal method