KRUSKAL-WALLIS TEST

- a rank-based nonparametric test
- purpose: to compare more than 2 populations

- **Example** (re Daniel text, p228. Cawson et al study of cortisol levels at time of delivery according to group type: group I – elective Caesarean section before onset of labor; group II – emergency Caesarean section during induced labor; group III – spontaneous labor.)

Group I:	262	307	211	323	454	339	304	154	287	356
Group II:	465	501	455	355	468	362				
Group III:	343	772	207	1048	838	687				

Test the following **hypotheses** using $\alpha = .05$

H_o: the 3 populations are identical
H_a: the 3 populations are not identical
(or the 3 populations do not have the same median assuming the 3 populations have same shape)

Ranks corresponding to the data

											Rank sums: R_i
Group I:	4	7	3	8	14	9	6	1	5	12	$R_1 = 69$
Group II:	16	18	15	11	17	13					$R_2 = 90$
Group III:	10	20	2	22	21	19					$R_3 = 94$

Test statistic: $T = \frac{12}{N(N+1)} \sum \frac{R_i^2}{n_i} - 3(N+1) \quad \dot{\sim} \quad \chi^2$ (2) distribution under H_o

Decision rule: Reject H_o if observed t $> \chi^2(2, .95) = 5.991$

Observed test stat. value: $t = \frac{12}{22(23)} (69^2/10 + 90^2/6 + 94^2/6) - 3(22+1) = 9.232$

P-value (approximate): Using TI-83, χ^2cdf(9.232, 100000, 2) = .0099

Conclusion: Since $9.232 > 5.991$, we reject H_o. There is sufficient statistical evidence to conclude that the populations are not identical. If the null hypothesis were true, the chances of observing a test statistic value as extreme as that obtained from the data are about 1 in a 100.

- <u>Distributional theory for test statistic</u>

Consider the sampling scheme where n integers are selected at random, without replacement, from the first N integers, 1 to N. Let X_i be the ith integer selected, and let

$$T_n = X_1 + X_2 + ... + X_n$$

be the sum of the integers selected. The expected value of T_n is given by

$$E(T_n) = \frac{n(N+1)}{2}$$

and the variance of T_n is given by

$$Var(T_n) = \frac{n(N+1)(N-n)}{12}.$$

(See W.J. Conover, *Practical Nonparametric Statistics*, 2nd Ed., 1980, Wiley, New York).

A version of the central limit theorem implies that $Z = \frac{T_n - E(T_n)}{\sqrt{VAR(T_n)}}$ has an approximate

standard normal distribution when n is of at least moderate size, say $n > 5$. In our case, replace T_n with R_i, the sum of the ranks for group i. Then

$$Z = \frac{R_i - E(R_i)}{\sqrt{VAR(R_i)}} = \frac{R_i - n_i(N+1)/2}{\sqrt{n_i(N+1)(N-n_i)/12}} \stackrel{\bullet}{\sim} N(0,1) .$$

And so $Z^2 = \frac{(R_i - n_i(N+1)/2)^2}{n_i(N+1)(N-n_i)/12} \stackrel{\bullet}{\sim} \chi^2(1)$. But, since the R_i's are not independent, an

adjustment is needed when summing, and one degree of freedom is lost. The weighted sum of the Z^2 's for all k groups is

$$T = \sum_{i=1}^{k} \frac{N-n_i}{N} \frac{(R_i - n_i(N+1)/2)^2}{n_i(N+1)(N-n_i)/12} \stackrel{\bullet}{\sim} \chi^2(k-1).$$

Now use algebra to show that

$$T = \sum_{i=1}^{k} \frac{N-n_i}{N} \frac{(R_i - n_i(N+1)/2)^2}{n_i(N+1)(N-n_i)/12} = \frac{12}{N(N+1)} \sum \frac{R_i^2}{n_i} - 3(N+1)$$

- For $k = 3$, $n_i \leq 5$, and no ties, use Table A8 in Conover to obtain exact quantiles.

- If there are many ties, and $n_i > 5$, use the test statistic

$$T(\text{ties}) = \frac{1}{S^2}\left(\sum \frac{R_i^2}{n_i} - \frac{N(N+1)^2}{4}\right) \text{ where}$$

$$S^2 = \frac{1}{N-1}\left(\sum_{\text{all ranks}} R(X_{ij})^2 - \frac{N(N+1)^2}{4}\right).$$

Under H_o, $T(\text{ties}) \dot{\sim} \chi^2(k-1)$.

- Multiple Comparisons: If H_o is rejected at level α, we can say that populations i and j seem to be different if the following inequality is satisfied

$$|\ R_i/n_i - R_j/n_j\ | > t_{1-\alpha/2}\left(S^2\ \frac{N-1-T}{N-k}\right)^{1/2}\left(\frac{1}{n_i} + \frac{1}{n_j}\right)^{1/2}.$$

example: comparing groups I and II

$$|\ 69/10 - 90/6\ | \overset{?}{>} 2.093\left(\frac{22(23)}{12}\ \frac{22-1-9.232}{19}\right)^{1/2}\left(\frac{1}{10} + \frac{1}{6}\right)^{1/2}$$

Since 8.1 $>$ 2.639, group I differs significantly from group II, statistically speaking.