Counting Integer Functions
with Preimage Size-2 Constraints

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For positive integer k let $[k] = \{1, 2, \ldots, k\}$ and let $[0] = \emptyset$. Consider a function $f : [2n] \to [n]$ such that the cardinality of $f^{-1}\{y\}$ is 2 for every $y \in [n]$. For example, $f : [4] \to [2]$ defined by $f(1) = 2,\ f(2) = 1, f(3) = 2,$ and $f(4) = 1$ is such a function. For given n, how many functions are there of this type?

Counting the number of ways to choose the elements of $f^{-1}\{1\}$, then to choose the elements of $f^{-1}\{2\}$, ..., and finally to choose the elements of $f^{-1}\{n\}$, we obtain $\binom{2n}{2}\binom{2n-2}{2} \cdot\ \cdot\ \cdot \binom{2}{2}$ which simplifies to $(2n)!/2^n$.

Let $a(n) = (2n)!/2^n$. This sequence appears in the *On Line Encyclopedia of Integer Sequences* as sequence A000680 with several interpretations. We note the following facts about $a(n)$:

i) In *Maple*, the first 9 terms in the sequence can be generated by the code

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> seq(product(binomial(2*n-2*k,2),k=0..n-1),n=0..8);
1, 1, 6, 90, 2520, 113400, 7484400, 681080400, 81729648000
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ii) A recursive definition is given by $a(n) = \binom{2n}{2} a(n-1)$ with $a(0) = 1$.

iii) Also, for even n,

$$a(n) = \binom{2n}{n} a(n/2) a(n/2)$$

and, for odd n,

$$a(n) = \binom{2n}{n+1} a(\tfrac{n+1}{2}) a(\tfrac{n-1}{2}).$$

The derivation of (iii) above is as follows. If we let $c(n, m)$ denote the cardinality of the set $\{f : [n] \to [m]$ where $|f^{-1}\{y\}|$ is 2 for every $y \in [m]\}$, then it can be shown that

$$c(n, m) = \sum_{k=0}^{n} \binom{n}{k} c(k, m_1) c(n-k, m_2)$$

for all nonnegative integers m_1 and m_2 such that $m_1 + m_2 = m$. Hence, for even n,

$$c(2n, n) = \binom{2n}{n} c(n, n/2) c(n, n/2)$$

or, equivalently,

$$a(n) = \binom{2n}{n} a(n/2) a(n/2). \qquad (1)$$

Also, for odd n,

$$c(2n, n) = \binom{2n}{n+1} c(n+1, \tfrac{n+1}{2}) c(n-1, \tfrac{n-1}{2})$$

or, equivalently,

$$a(n) = \binom{2n}{n+1} a(\tfrac{n+1}{2}) a(\tfrac{n-1}{2}). \qquad (2).$$