The Probability of a Tied Vote
under a Multiple Vote Scheme

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Suppose there are several candidates vying for two-at-large positions on an elected board, commission, or council. Suppose further that the candidates are all equally favored by the electorate, that is, any pair of candidates is equally likely to get the two votes of any voter. Under these assumptions, we derive the probability of a tied vote . Using Legendre polynomials and their generating function, we also derive the generating function for the number of outcomes resulting in such a tie.

Joint Mathematics Meetings, San Antonio, Texas
January 2005

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Motivation.

- Election in Franklin, Tennessee in which there were 6 candidates vying for 4 municipal seats.

- Each of 4522 voters cast 4 votes. Two candidates tied.

- Jim East, columnist for the *Franklin Review Appeal* wanted the odds of a tie occurring.

- Exact probability eluded me.

- Simulations provided an estimate of .099 $\pm$ .022 with 95% confidence.

- Decided to investigate easier problems.

Example. Election with 5 candidates and 6 voters resulting in a tied vote for candidates 1 and 2.

	C_1	C_2	C_3	C_4	C_5
V_1	1	0	0	1	0
V_2	0	1	1	0	0
V_3	1	1	0	0	0
V_4	0	0	1	0	1
V_5	1	0	1	0	0
V_6	0	1	0	0	1
	3	3	4	2	2

Question. How many different election results (such as that above) produce a tie between candidates 1 and 2?

Voter 1 (and 5) above cast a type A ballot.
Voter 2 (and 6) above cast a type B ballot.
Voter 3 above cast a type C ballot.
Voter 4 above cast a type D ballot.

Answer. Let N_i denote the number of votes for C_i, and let $T_r(n)$ denote the number of ordered ballots that result with $N_1 = N_2$ when there are r candidates and n voters. Then

$$\begin{aligned} T_5(6) &= \sum_{k=0}^{3} \binom{6}{k}\binom{6-k}{k} 3^{2k} \sum_{j=0}^{6-2k} \binom{6-2k}{j} \binom{3}{2}^{6-2k-j} \\ &= 204,436 \end{aligned}$$

Thus $P(N_1 = N_2) = .20436$.

I. In general, $T_r(n)$, number of ordered ballots with a tie for C_1 and C_2 when there are r candidates and n voters, is given by

$$T_r(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k}\binom{n-k}{k}(r-2)^{2k}\left(\binom{r-2}{2}+1\right)^{n-2k}.$$

Since each voter of the n voters must choose 2 of r candidates, the total number of possible election results is $\binom{r}{2}^{n}$. Therefore, the probability $P_{n,r}$ of a tie between C_1 and C_2 is given by

$$\boxed{P_{n,r} = \binom{r}{2}^{-n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k}\binom{n-k}{k}(r-2)^{2k}\left(\binom{r-2}{2}+1\right)^{n-2k}}.$$

II. The Generating Function for $T_n(r)$.

The generating function $g_r(t)$ for $T_r(n)$ is given below.

$$g_r(t) = \sum_{n=0}^{\infty} T_n(r)\, t^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k}\binom{n-k}{k}(r-2)^{2k}\left(\binom{r-2}{2}+1\right)^{n-2k} t^n$$

$\ldots$

$$= \left(1 - 2\left(\binom{r-2}{2}+1\right)t + \left(\left(\binom{r-2}{2}+1\right)^2 - 4(r-2)^2\right)t^2\right)^{-1/2}$$

or

$$= \frac{1}{\sqrt{\left(1-\binom{r}{2}t\right)\left(1+\left(\binom{r}{2}-2\left(\binom{r-2}{2}+1\right)\right)t\right)}}.$$

Example.

$$
\begin{aligned}
g_5(t) &= \sum_{n=0}^{\infty} T_n(5)\, t^n \\
&= \frac{1}{\sqrt{\left(1-10t\right)\left(1+2t\right)}} \\
&= 1 + 4t + 34t^2 + 280t^3 + 2470t^4 + 22264t^5 \\
&+ 204436t^6 + 1900336t^7 + 17830150t^8 + \ldots
\end{aligned}
$$

One-Vote Scheme with r Candidates

- When there are r candidates and n voters, we obtain the following result.

$$P(N_1 = N_2) = \left(\tfrac{1}{r}\right)^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k}\binom{n-k}{k}(r-2)^{n-2k},$$

where N_i denotes the number of votes for candidate i.

To see this, consider a set of ordered ballots with vote tally
$(k,\ k, n_3, \ldots, n_r)$. There are $\binom{n}{k}$ ways to choose the k voters that vote for candidate C_1, $\binom{n-k}{k}$ ways to choose the k voters that vote for candidate C_2, and $(r-2)^{n-2k}$ ways for the remaining $(n-2k)$ voters to pick one of the $(r-2)$ remaining candidates.

- Let $a(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k}\binom{n-k}{k}(r-2)^{n-2k}$.

The generating function for $a(n)$, $n = 0, 1, 2, \ldots$, is

$$g(t) = \sum_{n=0}^{\infty} a(n)\, t^n = \frac{1}{\sqrt{(1-rt)(1-(r-4)t)}}.$$

In which case, we could write

$$P(N_1 = N_2) = \left(\tfrac{1}{r}\right)^n \tfrac{1}{n!}\, D^{(n)} g(0).$$

Derivation of the Generating Function for $T_n(r)$.

Let $g_r(t) = \sum_{n=0}^{\infty} T_n(r)\, t^n$. Then

$$g_r(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \binom{n-k}{k} (r-2)^{2k} \left(\binom{r-2}{2} + 1\right)^{n-2k} t^n$$

$$= \sum_{n=0}^{\infty} \left(\left(\binom{r-2}{2} + 1\right)^2 - 4(r-2)^2\right)^{n/2} P_n\left(\frac{\binom{r-2}{2}+1}{\sqrt{\left(\binom{r-2}{2}+1\right)^2 - 4(r-2)^2}}\right) t^n$$

[where $P_n(x)$ denotes the Legendre polynomial of degree n]

$$= \sum_{n=0}^{\infty} P_n\left(\frac{\binom{r-2}{2}+1}{\sqrt{\left(\binom{r-2}{2}+1\right)^2 - 4(r-2)^2}}\right) \left(t \sqrt{\left(\binom{r-2}{2} + 1\right)^2 - 4(r-2)^2}\ \right)^n.$$

Using the fact that the generating function for Legendre polynomials is given by

$$\sum_{n=0}^{\infty} P_n(x) t^n = \left(1 + 2xt + t^2\right)^{-1/2}$$

[see H.W. Gould's *Combinatorial Identities*, p.38],

we obtain

$$g_r(t) = \left(1 + 2\left(\binom{r-2}{2} + 1\right)t + \left(\left(\binom{r-2}{2} + 1\right)^2 - 4(r-2)^2\right)t^2\right)^{-1/2} \qquad \square$$

The Probability of a Tie Vote in an Election When Candidates Are Equally Favored

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Suppose there are r candidates who are equally favored by the electorate. If the candidates are vying for one elected position and n random voters cast one vote each, what is the probability that two particular candidates receive the same number of votes?

Theorem 1. For $i = 1, ..., r$, let N_i denote the number of votes that candidate i receives. Then, for $i \neq j$, we have

$$P(N_i = N_j) = \left(\tfrac{1}{r}\right)^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k}\binom{n-k}{k}(r-2)^{n-2k} \ . \tag{1}$$

Proof. Without loss of generality, let $i = 1$ and $j = 2$. Consider the event where $N_1 = N_2 = k$, where k ranges from 0 to $\lfloor n/2 \rfloor$. There are $\binom{n}{k}$ ways to choose k of the n voters who vote for candidate 1, there are $\binom{n-k}{k}$ ways to choose k of the remaining $n-k$ voters who vote for candidate 2, and there are $(r-2)^{n-2k}$ ways for the final remaining $n-2k$ voters to choose a candidate other than candidates 1 and 2. Since each voter has r choices, there are r^n possible ordered ballots. Therefore, we have

$$\begin{aligned} P(N_i = N_j) &= \sum_{k=0}^{\lfloor n/2 \rfloor} P(N_1 = N_2 = k) \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{\binom{n}{k}\binom{n-k}{k}(r-2)^{n-2k}}{r^n} \\ &= \left(\tfrac{1}{r}\right)^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k}\binom{n-k}{k}(r-2)^{n-2k}. \qquad \square \end{aligned}$$

Let $a_r(n)$ denote the number of ordered ballots for which $N_1 = N_2$. As shown above,

$$a_r(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k}\binom{n-k}{k}(r-2)^{n-2k}. \tag{2}$$

We want to derive the generating function, g_r, for $a_r(n)$. But, first, we need the following lemma which involves Legendre polynomials. Let $p_n(x)$ denote the Legendre polynomial of degree n which can be defined by

$$p_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \text{ for } n = 0, 1, 2, \ldots .$$

***Lemma* 2.** The number $a_r(n)$ of ordered ballots for which N_1 equals N_2 can be expressed as follows:

$$a_r(n) = [(r-2)^2 - 4]^{n/2} p_n\left(\frac{r-2}{\sqrt{(r-2)^2-4}}\right)$$

where $p_n(x)$ denotes the Legendre polynomial of degree n.

***Proof*.** The following identity can be found in H.W. Gould's *Combinatorial Identities* [identity 3.137]:

$$p_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} 2^{-2k}\, x^{n-2k}(x^2-1)^k \ .$$

A little algebra done carefully leads to

$$p_n(x) = \tfrac{1}{2^n}\left(x^2-1\right)^{n/2} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k}\left(2x(x^2-1)^{-1/2}\right)^{n-2k}.$$

Letting $r-2 \ = \ 2x(x^2-1)^{-1/2}$, we get

$$x = \frac{r-2}{\sqrt{(r-2)^2-4}} \qquad \text{and} \qquad \tfrac{1}{2^n}\left(x^2-1\right)^{n/2} = [(r-2)^2-4]^{-n/2}.$$

Hence

$$\begin{aligned} p_n\left(\frac{r-2}{\sqrt{(r-2)^2-4}}\right) &= [(r-2)^2-4]^{-n/2} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k}(r-2)^{n-2k} \\ &= [(r-2)^2-4]^{-n/2}\, a_r(n). \end{aligned}$$

Upon solving for $a_r(n)$ we get the desired identity. □

The generating function L for $p_n(x)$ is well known and is given by

$$L(z) = \sum_{n=0}^{\infty} p_n(x)\, z^n = (1-2xz+z^2)^{-1/2}. \tag{3}$$

We can now find the generating function g_r for $a_r(n)$ as follows:

$$\begin{aligned}
g_r(t) &= \sum_{n=0}^{\infty} a_r(n)\, t^n \\
&= \sum_{n=0}^{\infty} [(r-2)^2-4]^{n/2} p_n\left(\tfrac{r-2}{\sqrt{(r-2)^2-4}}\right) t^n \\
&= \sum_{n=0}^{\infty} p_n\left(\tfrac{r-2}{\sqrt{(r-2)^2-4}}\right)\Big([(r-2)^2-4]^{1/2}\, t\Big)^n \\
&= (1-2(r-2)t+((r-2)^2-4)t^2)^{-1/2} \qquad \text{by (3)} \\
&= (1-2rt+4t+r^2t^2-4rt^2)^{-1/2} \\
&= \Big((1-rt)(1-(r-4)t)\Big)^{-1/2}.
\end{aligned}$$