

Evaluating at $t = a$, we see that $m(a)$ divides $k_1^n f(r(a))$. Any common prime divisor of $m(a)$ and k_1^n divides $k_2 b$, but k_1 is relatively prime to b by (2) and also to k_2 . Thus $m(a)$ is relatively prime to k_1 , so it divides $f(r(a))$, and we are done. ■

REFERENCES

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2. Joe Roberts, *Elementary Number Theory*, MIT Press, 1977, MR 58 #16472.

A Curious Way to Test for Primes

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Upon hearing that you are a student of mathematics, a cab driver says to you, “Check this out. The second derivative of e^x is e^x , right? And e^x evaluated at 0 is equal to 1, right? Therefore 2 has got to be a prime number.” Your first reaction is a condescending chuckle for the cabby who seems to dabble in mathematics and appears to make a jumble of it. “Well, at least she’s logically correct—2 is a prime number,” you mumble to yourself with some smugness. But the cab driver hears you and takes a long route to your destination. You end up paying a hefty fare. The cab driver smirks as she drives off.

In fact, the smirking cabby stated a specific case of the theorem we provide below, a theorem which offers an unusual characterization of prime numbers based on differentiation. The cab driver could give another specific case by stating, “The third derivative of $e^x + e^{x^2/2}$ evaluated at $x = 0$ is 1, and thus 3 is a prime number” Offering another example, she could state accurately that, “The fourth derivative of $e^x + e^{x^2/2} + e^{x^3/3}$ evaluated at 0 does not equal 1, and hence 4 is not a prime number.” You can probably guess the pattern. We give it below in a theorem with a surprisingly simple proof that uses the series expansion of $e^{x^k/k}$ and the power rule for differentiation.

THEOREM. For each positive integer $n > 1$, define the function g_n by $g_n(x) = \sum_{k=1}^{n-1} e^{x^k/k}$. A positive integer n is prime if and only if $\frac{d^n}{dx^n} g_n(0) = 1$.

Proof. Let n be a positive integer greater than 1. Note that $e^{x^k/k}$ has a series expansion given by

$$e^{x^k/k} = \sum_{j=0}^{\infty} \frac{(x^k/k)^j}{j!} = \sum_{j=0}^{\infty} \frac{x^{kj}}{k^j j!}$$

for all real x . Hence, we have

$$g_n(x) = \sum_{k=1}^{n-1} \sum_{j=0}^{\infty} \frac{x^{kj}}{k^j j!},$$

and upon differentiating g_n we obtain

$$\begin{aligned} \frac{d^n}{dx^n} g_n(x) &= \frac{d^n}{dx^n} \sum_{k=1}^{n-1} \sum_{j=0}^{\infty} \frac{x^{kj}}{k^j j!} \\ &= \sum_{k=1}^{n-1} \sum_{j=0}^{\infty} \frac{1}{k^j j!} \frac{d^n}{dx^n} (x^{kj}) \\ &= \sum_{k=1}^{n-1} \sum_{j=0}^{\infty} \frac{(kj)(kj-1)(kj-2) \cdots (kj-n+1)x^{kj-n}}{k^j j!} I[kj \geq n], \end{aligned}$$

where the indicator function $I[s]$ takes the value 1 if statement s is true and the value 0 otherwise. Upon evaluating $\frac{d^n}{dx^n} g_n(x)$ at $x = 0$, we see that every term in the inner sum above vanishes except for the terms where $kj = n$. Thus, using $k|n$ to denote the statement “ k divides n ,” we get

$$\begin{aligned} \frac{d^n}{dx^n} g_n(0) &= \sum_{k=1}^{n-1} \frac{n(n-1)(n-2) \cdots (n-n+1)}{k^{n/k} (n/k)!} I[k|n] \\ &= \sum_{k=1}^{n-1} \frac{n!}{k^{n/k} (n/k)!} I[k|n]. \end{aligned}$$

If n is prime, the only divisor of n that is less than or equal to $n - 1$ is 1, in which case the summation above collapses to the single term for $k = 1$. Hence, when n is prime, we obtain

$$\frac{d^n}{dx^n} g_n(0) = \frac{n!}{1^{n/1} (n/1)!} = 1.$$

If n is not prime and $n > 1$, there exist positive integers k and r , both in the interval $[2, n - 1]$, such that $n = kr$. Thus, if n is not prime and $n > 1$, we have

$$\frac{d^n}{dx^n} g_n(0) \geq 1 + \frac{(kr)!}{k^r r!} > 1.$$

We conclude with *Maple* code below that performs the primality test for small n .

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eval(diff(sum(exp(t^k/k), k = 1..n - 1), t$n), t = 0);
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Replace n in the code with the specific positive integer you wish to test. If the output is 1, then n is a prime. If the output is not 1, then is n not a prime.