1. $Y(t) = K(t)^{\alpha}[A(t)L(t)]^{1-\alpha}$ $\qquad \alpha \in (0,1) \qquad \delta \in (0,1)$

$\dot{K}(t) = sY(t) - \delta K(t)$ $\qquad \dot{A}(t) = gA(t) \qquad g \in (0,1)$

$\dot{L}(t) = nL(t) \qquad n \in (0,1)$

intensive form:

$$\hookrightarrow \quad \frac{Y(t)}{A(t)L(t)} = \left(\frac{K(t)}{A(t)L(t)}\right)^{\alpha}$$

$$y(t) = k(t)^{\alpha}$$

a) $\dot{k}(t) = sk(t)^{\alpha} - (\delta + n + g)k(t)$

on the balanced growth path, $\dot{k}(t) = 0$

$\Rightarrow \quad sk^{\alpha} = (\delta + n + g)k$

$$\begin{cases} k^* = \left(\frac{s}{\delta+n+g}\right)^{\frac{1}{1-\alpha}} \\ y^* = \left(\frac{s}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}} \\ c^* = (1-s)\left(\frac{s}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}} \end{cases}$$

b) $f'(k) = \delta + n + g = MPK$

$\Rightarrow \quad \alpha k^{\alpha-1} = \delta + n + g$

$$k^* = \left(\frac{\alpha}{\delta+n+g}\right)^{\frac{1}{1-\alpha}}$$

c) $\left(\frac{\alpha}{\delta+n+g}\right)^{\frac{1}{1-\alpha}} = \left(\frac{s}{\delta+n+g}\right)^{\frac{1}{1-\alpha}}$ $\longrightarrow$ with a Cobb-Douglas production function, the saving rate needed to reach the golden rule is equal to the elasticity of output with respect to capital

$\Rightarrow \quad \boxed{s = \alpha}$

2. $Y = \left[a(bK)^{\sigma} + (1-a)\{(1-b)\cdot AL\}^{\sigma} \right]^{\frac{1}{\sigma}}$

$a \in (0,1)$
$b \in (0,1)$
$\sigma < 1$

a) $F(\lambda K, \lambda AL) = \left[a(b\lambda K)^{\sigma} + (1-a)((1-b)\lambda AL)^{\sigma} \right]^{\frac{1}{\sigma}}$

$= \left\{ \lambda^{\sigma} \left[a(bK)^{\sigma} + (1-a)((1-b)AL)^{\sigma} \right] \right\}^{\frac{1}{\sigma}}$

$= \lambda \left[a(bK)^{\sigma} + (1-a)((1-b)AL)^{\sigma} \right]^{\frac{1}{\sigma}}$

$= \lambda F(K, AL)$

b) $y = \frac{Y}{AL} = \frac{1}{AL}\left[a(bK)^{\sigma} + (1-a)((1-b)AL)^{\sigma} \right]^{\frac{1}{\sigma}}$

$= \left\{ \left(\frac{1}{AL}\right)^{\sigma} \left[a(bK)^{\sigma} + (1-a)((1-b)AL)^{\sigma} \right] \right\}^{\frac{1}{\sigma}}$

$= \left[a\left(b\frac{K}{AL}\right)^{\sigma} + (1-a)\left((1-b)\frac{AL}{AL}\right)^{\sigma} \right]^{\frac{1}{\sigma}}$

$= \left[a(bk)^{\sigma} + (1-a)(1-b)^{\sigma} \right]^{\frac{1}{\sigma}} = f(k)$

c) $f'(k) = \cancel{\frac{1}{\sigma}}\left[a(bk)^{\sigma} + (1-a)(1-b)^{\sigma} \right]^{\frac{1-\sigma}{\sigma}} \cdot a\cancel{\sigma}(bk)^{\sigma-1}\cdot b$

$= ab^{\sigma}\left[a(bk)^{\sigma} + (1-a)(1-b)^{\sigma} \right]^{\frac{1-\sigma}{\sigma}} \cdot k^{\sigma-1}$

$= ab^{\sigma}\left[a(b\cancel{k})^{\sigma}\cancel{k^{-\sigma}} + (1-a)(1-b)^{\sigma}k^{-\sigma} \right]^{\frac{1-\sigma}{\sigma}}$

$= ab^{\sigma}\left[ab^{\sigma} + (1-a)(1-b)^{\sigma}\left(\frac{1}{k}\right)^{\sigma} \right]^{\frac{1-\sigma}{\sigma}}$

d) $\sigma \in (0,1)$:

$\lim_{k \to 0} f'(k) = ab^{\sigma}\left[ab^{\sigma} + (1-a)(1-b)^{\sigma}\left(\frac{1}{0^+}\right)^{\sigma} \right]^{\frac{1-\sigma}{\sigma}} = +\infty$

$\lim_{k \to \infty} f'(k) = ab^{\sigma}\left[ab^{\sigma} + (1-a)(1-b)^{\sigma}\left(\frac{1}{\infty}\right)^{\sigma} \right]^{\frac{1-\sigma}{\sigma}}$

$= ab^{\sigma}\left[ab^{\sigma} + 0 \right]^{\frac{1-\sigma}{\sigma}} = ab^{\sigma} a^{\frac{1-\sigma}{\sigma}} b^{1-\sigma}$

$= a^{1/\sigma} b > 0$

e) $q < 0$:

$$\lim_{k \to 0} f'(k) = ab^{q}\left[ab^{q} + (1-a)(1-b)^{q}\left(\frac{1}{0^{+}}\right)^{q}\right]^{\frac{1-q}{q}}$$

$$= ab^{q}\left[ab^{q} + 0\right]^{\frac{1-q}{q}} = ab^{q}\, a^{\frac{1-q}{q}}\, b^{1-q}$$

$$= a^{1/q}\, b < \infty$$

$$\lim_{k \to \infty} f'(k) = ab^{q}\left[ab^{q} + (1-a)(1-b)^{q}\left(\frac{1}{\infty}\right)^{q}\right]^{\frac{1-q}{q}}$$

$$= ab^{q}\left[ab^{q} + (1-a)(1-b)\left(0^{+}\right)^{q}\right]^{\frac{1-q}{q}}$$

$$= ab^{q}\left[ab^{q} + \infty\right]^{\frac{1-q}{q}}$$

$$= 0$$

3. $Y(t) = F\big(K(t), A(t)L(t)\big)$

$$\frac{\partial F\big(K(t), A(t)L(t)\big)}{\partial L(t)} = w(t) \rightarrow \text{"real wage" or the return to labor}$$

$$\frac{\partial F\big(K(t), A(t)L(t)\big)}{\partial K(t)} - \delta = r(t) \rightarrow \text{return to capital}$$

implied by constant returns to scale

a) $F\big(K(t), A(t)L(t)\big) = A(t)L(t)f(k(t))$

$$\frac{\partial F(\cdot)}{\partial L(t)} = A(t)\left[f(k(t)) + \cancel{L(t)}\, f'(k(t)) \cdot \frac{-K(t)}{(A(t)L(t))^2} \cdot \cancel{A(t)}\right]$$

$$= A(t)\left[f(k(t)) - \frac{K(t)}{A(t)L(t)} \cdot f'(k(t))\right]$$

$$= A(t)\big[f(k(t)) - k(t)f'(k(t))\big] = w(t)$$

b) $$\frac{\partial F(\cdot)}{\partial K(t)} = \cancel{A(t)L(t)}\, f'(k(t)) \cdot \frac{1}{\cancel{A(t)L(t)}} = f'(k(t)) = r(t) + \delta$$

$$\Rightarrow w(t)L(t) + r(t)K(t) = A(t)\big[f(k(t)) - k(t)f'(k(t))\big]L(t) + \big[f'(k(t)) - \delta\big]K(t)$$

$$= \underbrace{A(t)L(t)f(k(t))}_{F(\cdot)} - \underbrace{A(t)L(t)k(t)}_{K(t)} f'(k(t)) + K(t)f'(k(t)) - \delta K(t)$$

$$= F\big(K(t), A(t)L(t)\big) - \cancel{K(t)f'(k(t)) + K(t)f'(k(t))} - \delta K(t)$$

$$= F\big(K(t), A(t)L(t)\big) - \delta K(t)$$

c) $r(t) = f'(k(t)) - \delta$

on a balanced growth path $k(t) = k^*$

$\Rightarrow \quad r(t) = r^* = f'(k^*) - \delta$

so $r(t)$ is constant on a balanced growth path

$$\gamma_r \equiv \frac{\dot{r}(t)}{r(t)} = \frac{f''(k(t))\,\dot{k}(t)}{f'(k(t)) - \delta} \longrightarrow \text{growth rate of the return to capital}$$

on a balanced growth path $\dot{k}(t) \equiv 0$ and $k(t) = k^*$

$$\Rightarrow \quad \gamma_r = \frac{f''(k^*)\cdot 0}{f'(k^*) - \delta} = 0$$

↳ ★ the return to capital is constant on a balanced growth path

$$w(t) = A(t)\left[f(k(t)) - k(t)f'(k(t))\right]$$

$$\dot{w}(t) = \dot{A}(t)\left[f(k(t)) - k(t)f'(k(t))\right] + A(t)\left[\cancel{f'(k(t))\dot{k}(t)} - \cancel{\dot{k}(t)f'(k(t))} - k(t)f''(k(t))\dot{k}(t)\right]$$

$$= \dot{A}(t)\left[f(k(t)) - k(t)f'(k(t))\right] - A(t)k(t)f''(k(t))\dot{k}(t)$$

$$\gamma_w \equiv \frac{\dot{w}(t)}{w(t)} = \frac{\dot{A}(t)\cancel{\left[f(k(t)) - k(t)f'(k(t))\right]}}{A(t)\cancel{\left[f(k(t)) - k(t)f'(k(t))\right]}} - \frac{\cancel{A(t)}k(t)f''(k(t))\dot{k}(t)}{\cancel{A(t)}\left[f(k(t)) - k(t)f'(k(t))\right]}$$

$$= g - \frac{k(t)f''(k(t))}{f(k(t)) - k(t)f'(k(t))}\cdot \dot{k}(t)$$

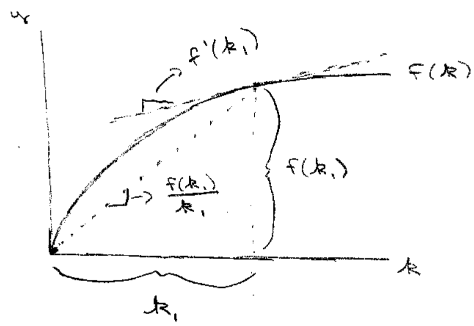
on a balanced growth path $\dot{k}(t) \equiv 0$ and $k(t) = k^*$

$$\Rightarrow \quad \gamma_w = g - \frac{k^*f''(k^*)}{f(k^*) - k^*f'(k^*)}\cdot 0 = g$$

↳ on a balanced growth path the real wage
★ grows at the rate of technological growth

d) $$\gamma_w = g - \frac{k(t)\, f''(k(t))}{f(k(t)) - k(t) f'(k(t))}\, \dot{k}(t)$$

$$= \underbrace{g}_{(+)} - \frac{k(t)}{k(t)} \cdot \frac{\overset{(-)}{f''(k(t))}}{\underbrace{\frac{f(k(t))}{k(t)} - f'(k(t))}_{?}}\, \overset{(+)}{\dot{k}(t)}$$ when $k(t) < k^*$



b/c f is a concave function, $\frac{f(k)}{k} > f'(k) \quad \forall k > 0$

$\Rightarrow \frac{f(k)}{k} - f'(k) > 0 \quad \forall k > 0$

$$\therefore \quad \gamma_w = \underbrace{g}_{(+)} - \frac{\overbrace{f''(k(t))}^{(-)}}{\underbrace{\frac{f(k(t))}{k(t)} - f'(k(t))}_{(+)}}\, \underbrace{\dot{k}(t)}_{(+)} > g$$

So when $k < k^*$ (ie $\dot{k}(t) > 0$) the growth rate of the real wage exceeds g. However, the growth rate γ_w gradually declines as k moves toward k^* and it stops falling once $\gamma_w = g$, or when the economy has reached a balanced growth path.

4. $Y(t) = K(t)^{\alpha} \left[A(t) L(t) \right]^{\beta} R^{1-\alpha-\beta}$ $\quad \alpha, \beta \in (0,1)$

$\alpha + \beta < 1$

$L'(t) = n L(t)$

$A'(t) = g A(t)$

$K'(t) = s Y(t) - \delta K(t)$

$R'(t) = 0$

$\hookrightarrow$ $Y(t) = K(t)^{1-\beta} K(t)^{\beta-1} K(t)^{\alpha} \left[A(t) L(t) \right]^{\beta} R^{1-\alpha-\beta}$

$= K(t)^{1-\beta} K(t)^{-(1-\alpha-\beta)} R^{1-\alpha-\beta} \left[A(t) L(t) \right]^{\beta}$

$= K(t)^{1-\beta} \left[\left\{ \left(\frac{R}{K(t)} \right)^{\frac{1-\alpha-\beta}{\beta}} A(t) \right\} L(t) \right]^{\beta}$

define $X(t) \equiv A(t) \left(\frac{R}{K(t)} \right)^{\frac{1-\alpha-\beta}{\beta}}$

$\Rightarrow$ $Y(t) = K(t)^{1-\beta} \left[X(t) L(t) \right]^{\beta}$

Intensive form:

$k(t) \equiv \frac{K(t)}{X(t) L(t)}$

$y(t) = \frac{Y(t)}{X(t) L(t)}$ $\Rightarrow$ $\boxed{y(t) = k(t)^{1-\beta}}$

Fundamental equation:

$$\dot{k}(t) = \frac{\dot{K}(t)}{X(t)L(t)} - \frac{K(t)}{\left(X(t)L(t)\right)^2}\left\{\dot{X}(t)L(t) + X(t)\dot{L}(t)\right\}$$

$$= s\frac{Y(t)}{X(t)L(t)} - \delta\frac{K(t)}{X(t)L(t)} - \frac{K(t)}{X(t)L(t)}\left\{\frac{\dot{X}(t)}{X(t)} + \frac{\dot{L}(t)}{L(t)}\right\}$$

$$= s\,y(t) - \delta k(t) - k(t)\left[n + \dot{X}(t)/X(t)\right]$$

$$= s\,k(t)^{1-\beta} - \left(\delta + n + \dot{X}(t)/X(t)\right)k(t)$$

$$\frac{\dot{X}(t)}{X(t)} = \frac{\dot{A}(t)\left(R/K(t)\right)^{\frac{1-\alpha-\beta}{\beta}}}{\dot{A}(t)\left(R/K(t)\right)^{\frac{1-\alpha-\beta}{\beta}}} - \frac{A(t)\frac{1-\alpha-\beta}{\beta}\left(R/K(t)\right)^{\frac{1-\alpha-\beta}{\beta}-1}\left(R/K(t)^2\right)\dot{K}(t)}{A(t)\left(R/K(t)\right)^{\frac{1-\alpha-\beta}{\beta}}}$$

$$= g - \frac{1-\alpha-\beta}{\beta}\left(R/K(t)\right)^{-1}\left(R/K(t)\right)\frac{\dot{K}(t)}{K(t)}$$

$$= g - \frac{1-\alpha-\beta}{\beta}\,\frac{\dot{K}(t)}{K(t)}$$

$$\dot{k}(t) = s\,k(t)^{1-\beta} - \left(\delta + n + g - \frac{1-\alpha-\beta}{\beta}\cdot\frac{\dot{K}(t)}{K(t)}\right)k(t)$$

$$= s\,k(t)^{1-\beta} - \left[\delta + n + g - \frac{1-\alpha-\beta}{\beta}s\,\frac{Y(t)}{K(t)} + \frac{1-\alpha-\beta}{\beta}\delta\,\frac{K(t)}{K(t)}\right]k(t)$$

$$= s\,k(t)^{1-\beta} - \left[\frac{1-\alpha}{\beta}\delta + n + g\right]k(t) + \frac{1-\alpha-\beta}{\beta}s\,\frac{Y(t)}{K(t)}\cdot\frac{K(t)}{X(t)L(t)}$$

$$= s\,k(t)^{1-\beta} - \left[\frac{1-\alpha}{\beta}\delta + n + g\right]k(t) + \frac{1-\alpha-\beta}{\beta}s\,y(t)$$

$$\Longrightarrow \boxed{\dot{k}(t) = \frac{1-\alpha}{\beta}s\,k(t)^{1-\beta} - \left[\frac{1-\alpha}{\beta}\delta + n + g\right]k(t)}$$

steady state:

$$\dot{k}(t) = 0$$

$$\hookrightarrow \quad \frac{1-\alpha}{\beta} s k^{1-\beta} = \left(\frac{1-\alpha}{\beta}\delta + n + g\right) k$$

$$\left(\frac{\frac{1-\alpha}{\beta} s}{\frac{1-\alpha}{\beta}\delta + n + g}\right) = k^{\beta}$$

$$\boxed{k^* = \left(\frac{\frac{1-\alpha}{\beta} s}{\frac{1-\alpha}{\beta}\delta + n + g}\right)^{1/\beta}}$$

balanced growth:

$$K(t) = X(t)L(t)k^*$$

$$\gamma_K \equiv \frac{\dot{K}(t)}{K(t)} = \frac{\dot{X}(t)}{X(t)} + n$$

$$= g - \frac{1-\alpha-\beta}{\beta}\frac{\dot{K}(t)}{K(t)} + n$$

$$\frac{1-\alpha}{\beta}\gamma_K = n+g$$

$$\boxed{\gamma_K = \frac{\beta}{1-\alpha}(n+g)}$$

$$Y(t) = X(t)L(t)y^*$$

$$\gamma_Y \equiv \frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{X}(t)}{X(t)} + n$$

$$= g - \frac{1-\alpha-\beta}{\beta}\gamma_K + n$$

$$= n+g - \frac{1-\alpha-\beta}{\cancel{\beta}}\frac{\cancel{\beta}}{1-\alpha}(n+g)$$

$$= (n+g) - \frac{1-\alpha-\beta}{1-\alpha}(n+g)$$

$$\boxed{\gamma_Y = \frac{\beta}{1-\alpha}(n+g)}$$

$$\hookrightarrow \quad \alpha+\beta < 1$$

$$\beta < 1-\alpha$$

$$\frac{\beta}{1-\alpha} < 1$$

$$\therefore \quad \gamma_K = \gamma_Y < n+g$$

The Malthusian assumption does not preclude growth. Although, the balanced growth rates are smaller than the ones order the standard Solow model w/ constant returns to scale