1. $U_t = \log c_{1t} + \frac{1}{1+\rho} \log c_{2t+1}$ $\qquad \rho > 0$

$Y_t = K_t^{\alpha} \left[A_t L_t \right]^{1-\alpha}$ $\qquad \alpha \in (0,1)$

$L_t = (1+n) L_{t-1}$

$A_t = (1+n) A_{t-1}$

a)

$$\left.\begin{array}{l} c_{1t} + S_t = w_t \\ c_{2t+1} = (1+r_{t+1}) S_t \end{array}\right) \longrightarrow \quad c_{2t+1} = (1+r_{t+1})(w_t - c_{1t})$$

$$\boxed{c_{1t} + \frac{1}{1+r_{t+1}} c_{2t+1} = w_t}$$

$$\max_{c_{1t},\, c_{2t+1}} \quad \mathcal{L} = \log c_{1t} + \frac{1}{1+\rho} \log c_{2t+1} + \lambda \left[w_t - c_{1t} - (1+r_{t+1})^{-1} c_{2t+1} \right]$$

$$\left.\begin{array}{ll} c_{1t}: & \frac{1}{c_{1t}} = \lambda \\ c_{2t+1}: & \frac{1}{1+\rho} \frac{1}{c_{2t+1}} = \frac{1}{1+r_{t+1}} \lambda \end{array}\right) \longrightarrow \quad \left(\frac{1+r_{t+1}}{1+\rho} \right) = \left(\frac{c_{2t+1}}{c_{1t}} \right)$$

$$\Longrightarrow \quad c_{1t} + \frac{1}{\cancel{1+r_{t+1}}} \cdot \underbrace{\left(\frac{\cancel{1+r_{t+1}}}{1+\rho} \right) c_{1t}}_{c_{2t+1}} = w_t$$

$$\left(\frac{2+\rho}{1+\rho} \right) c_{1t} = w_t$$

$$\boxed{c_{1t} = \left(\frac{1+\rho}{2+\rho} \right) w_t} \qquad \boxed{c_{2t+1} = \left(\frac{1+r_{t+1}}{2+\rho} \right) w_t}$$

$$\boxed{S_t = \left(\frac{1}{2+\rho} \right) w_t}$$

b) $$\frac{C_{2t+1}}{C_{1t}} = \frac{1+r_{t+1}}{1+\rho}$$

$\Rightarrow$ consumption rises w/ age if $r_{t+1} > \rho$

in other words, if the return on saving exceeds the discount rate, households will opt to save more for future consumption.

$\Rightarrow$ consumption falls w/ age if $r_{t+1} < \rho$

c) $K_{t+1} = S_t L_t \longrightarrow$ describes the evolution of the aggregate capital stock derived in class.

$$k_{t+1} = \frac{K_{t+1}}{A_{t+1}L_{t+1}} = \frac{S_t L_t}{A_{t+1}L_{t+1}} = \frac{S_t}{A_{t+1}} \cdot (1+n)^{-1} = \left(\frac{1}{2+\rho}\right) w_t \frac{1}{A_t} \cdot \frac{A_t}{A_{t+1}} (1+n)^{-1}$$

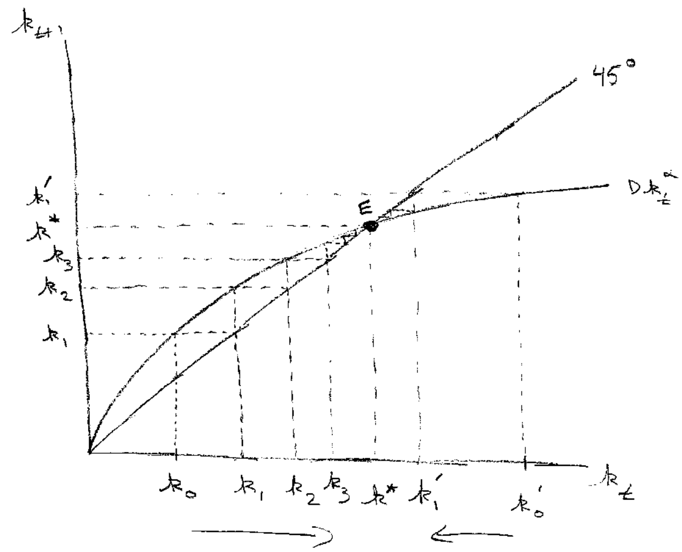
$$= \left(\frac{1}{2+\rho}\right) \frac{w_t}{A_t} (1+n)^{-1}(1+g)^{-1} = \left(\frac{1}{2+\rho}\right)\left[f(k_t) - k_t f'(k_t)\right](1+n)^{-1}(1+g)^{-1}$$

$$\boxed{k_{t+1} = \underbrace{\frac{1}{(1+n)(1+g)} \cdot \left(\frac{1}{2+\rho}\right)(1-\alpha)}_{D} k_t^{\alpha}} = D k_t^{\alpha}$$

balanced growth: $k_{t+1} = k_t = k^* \;\; \forall t$

$$k^* = \left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\right]^{\frac{1}{1-\alpha}}$$

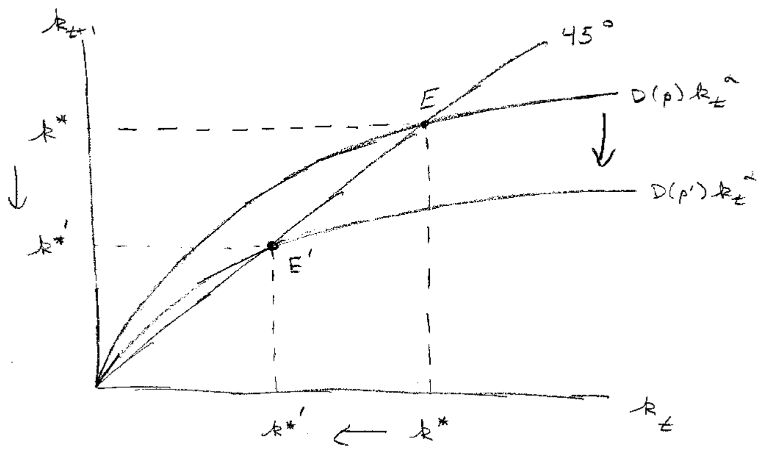
$$y^* = \left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\right]^{\frac{\alpha}{1-\alpha}}$$



Regardless of the initial value of k, the economy converges to a stable equilibrium k^* at point E provided $k_0 > 0$. The slope of the function describing the evolution of k_t is less than one at the point of intersection w/ the 45° line. Because it intersects the 45° line only once, E is a unique stable equilibrium.

d) $k_{t+1} = D k_t^{\alpha}$ where $D = \frac{1-\alpha}{(1+n)(1+g)(2+\rho)}$

$\frac{\partial D}{\partial \rho} < 0 \implies$ an increase in ρ shifts the $D k_t^{\alpha}$ schedule down!



$\therefore \frac{\partial k^*}{\partial \rho} < 0$ An increase in the discount rate makes households value future consumption less. They will therefore reduce saving $\Rightarrow$ reduction in capital accumulation

$$y^* = \left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\right]^{\frac{\alpha}{1-\alpha}}$$

$$\hookrightarrow \frac{\partial y^*}{\partial \rho} = \frac{\alpha}{1-\alpha}\left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\right]^{\frac{\alpha}{1-\alpha}-1} \cdot \left[\frac{-(1-\alpha)}{(1+n)^2(1+g)^2(2+\rho)^2}\right] \cdot (1+n)(1+g)$$

$$= -\frac{\alpha}{1-\alpha}\left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\right]^{\frac{\alpha}{1-\alpha}-1} \cdot \left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\right] \cdot \frac{1}{2+\rho}$$

$$= -\frac{\alpha}{1-\alpha}\left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\right]^{\frac{\alpha}{1-\alpha}} \cdot \frac{1}{2+\rho} = -\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{2+\rho}\right) y^* < 0$$

$$\implies \frac{\rho}{y^*} \cdot \frac{\partial y^*}{\partial \rho} = \frac{\rho}{\cancel{y^*}} \cdot \left(-\frac{\alpha}{1-\alpha}\right) \cdot \left(\frac{1}{2+\rho}\right) \cancel{y^*} = -\frac{\alpha \rho}{(1-\alpha)(2+\rho)} < 0$$

2.14

$$Y_t = F(K_t, A_t L_t)$$

$$K_{t+1} = K_t + sY_t - \delta K_t$$

$$L_{t+1} = (1+n) L_t$$

$$A_{t+1} = (1+g) A_t$$

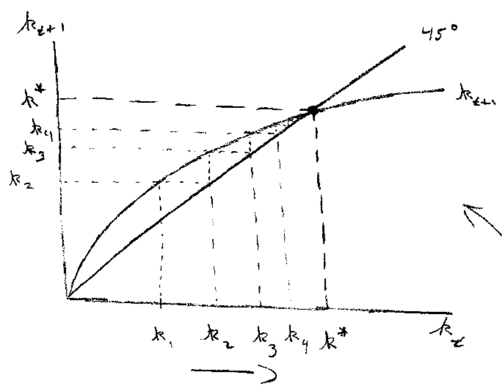
a) $k_t \equiv \dfrac{K_t}{A_t L_t}$ $\qquad y_t \equiv \dfrac{Y_t}{A_t L_t}$ $\qquad y_t = f(k_t)$

$$\frac{K_{t+1}}{A_{t+1} L_{t+1}} \cdot \frac{A_{t+1} L_{t+1}}{A_t L_t} = \frac{K_t}{A_t L_t} + s\frac{Y_t}{A_t L_t} - \delta \frac{K_t}{A_t L_t}$$

$$k_{t+1} \cdot (1+g)(1+n) = k_t + s f(k_t) - \delta k_t$$

$$* \quad k_{t+1} = \frac{s}{(1+n)(1+g)} f(k_t) + \frac{1-\delta}{(1+n)(1+g)} k_t$$

b)



proof

$$\frac{\partial k_{t+1}}{\partial k_t} = \frac{s}{(1+n)(1+g)} f'(k_t) + \frac{1-\delta}{(1+n)(1+g)} > 0$$

$$\frac{\partial^2 k_{t+1}}{\partial k_t^2} = \frac{s}{(1+n)(1+g)} f''(k_t) < 0$$

$$\lim_{k \to 0} \frac{\partial k_{t+1}}{\partial k_t} = \infty$$

$$\lim_{k \to \infty} \frac{\partial k_{t+1}}{\partial k_t} = \frac{1-\delta}{(1+n)(1+g)} \in (0,1)$$

$$\lim_{t \to \infty} k_t = k^*$$

c) $c_t = \frac{C_t}{A_t L_t} = (1-s)\frac{Y_t}{A_t L_t}$

$c_t = (1-s)\, y_t$

$\hookrightarrow \quad k_t = k_{t+1} = k^*$

$k^* = \frac{s}{(1+n)(1+g)} f(k^*) + \frac{1-\delta}{(1+n)(1+g)} k^*$

$\{(1+n)(1+g) - 1 + \delta\} k^* = s f(k^*)$

$k^*(\delta + n + g + ng) = s f(k^*)$

$\Rightarrow \quad c^* = \left(1 - (\delta+n+g+ng)\, k^*/f(k^*)\right) f(k^*)$

$* \; c^* = f(k^*) - (\delta+n+g+ng)\, k^*$.

$\max_{k^*} \; c(k^*)$

$k^* : \; f'(k^*) = \delta + n + g + ng$.

d) $f(k) = k^\alpha$

(i) $k_{t+1} = \frac{s}{(1+n)(1+g)} k_t^\alpha + \frac{1-\delta}{(1+n)(1+g)} k_t$.

(ii) $k^*(\delta+n+g+ng) = s\, k^{*\alpha}$

$k^* = \left(\frac{s}{\delta+n+g+ng}\right)^{1/1-\alpha}$.

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4. $\max_{c_{1t}, c_{2t+1}, s_t} \quad u(c_{1t}) + \beta u(c_{2t+1}) \quad \text{st}$

$$c_{1t} + s_t \le w_t$$

$$c_{2t+1} \le w_{t+1} + (1+r_{t+1}) s_t$$

$u'(c) > 0$, $u''(c) < 0$ $\forall c$

a. $\max_{s_t} \quad u(w_t - s_t) + \beta u(w_{t+1} + (1+r_{t+1}) s_t)$

$$s_t : \quad -u'(w_t - s_t) + \beta u'(w_{t+1} + (1+r_{t+1}) s_t)(1+r_{t+1}) = 0$$

Because the utility function is increasing and strictly concave, the Euler eqn is sufficient to characterize an optimal saving plan given market prices. The implicit function theorem implies $\exists$

$$s_t = s(w_t, w_{t+1}, r_{t+1}) \quad \text{where} \quad s: \mathbb{R}^3_+ \longrightarrow \mathbb{R}$$

b. $$-u''(c_{1t})(1 - \partial s_t/\partial w_t) + \beta u''(c_{2t+1})(1+r_{t+1})^2 \, \partial s_t/\partial w_t = 0$$

$$\frac{\partial s_t}{\partial w_t} = \frac{u''(c_{1t})}{u''(c_{1t}) + \beta(1+r_{t+1})^2 u''(c_{2t+1})} \in (0,1)$$

c. $$u''(c_{1t}) \, \partial s_t/\partial w_{t+1} + \beta u''(c_{2t+1})\left(1 + (1+r_{t+1}) \, \partial s_t/\partial w_{t+1}\right)(1+r_{t+1}) = 0$$

$$\frac{\partial s_t}{\partial w_{t+1}} = \frac{-\beta u''(c_{2t+1})(1+r_{t+1})}{u''(c_{1t}) + \beta(1+r_{t+1})^2 u''(c_{2t+1})} < 0$$

d. suppose $u(c) = \dfrac{c^{1-\theta}}{1-\theta}$

$$\Rightarrow \beta\left[w_{t+1} + (1+r_{t+1}) s_t\right]^{-\theta}(1+r_{t+1}) = (w_t - s_t)^{-\theta}$$

$$\beta^{-1/\theta}(1+r_{t+1})^{-1/\theta}\left[w_{t+1} + (1+r_{t+1}) s_t\right] = w_t - s_t$$

$$\left[(1+r_{t+1})^{\frac{\theta-1}{\theta}} \beta^{-1/\theta} + 1\right] s_t = w_t - w_{t+1}(1+r_{t+1})^{-1/\theta} \beta^{-1/\theta}$$

$$\Rightarrow \left[(1+r_{t+1})^{\frac{\theta-1}{\theta}}\rho^{-1/\theta}+1\right]\frac{\partial s_t}{\partial r_{t+1}} + \left(\frac{\theta-1}{\theta}\right)(1+r_{t+1})^{\frac{\theta-1}{\theta}-1}\rho^{-1/\theta} s_t = \frac{1}{\theta} w_{t+1}(1+r_{t+1})^{-1/\theta-1}\rho^{-1/\theta}$$

define $\Psi_{t+1} \equiv 1+\rho^{-1/\theta}(1+r_{t+1})^{\theta-1/\theta} > 1$

$$\frac{\partial s_t}{\partial r_{t+1}} = \left(\frac{1-\theta}{\theta}\right)\left[\rho(1+r_{t+1})\right]^{-1/\theta}\frac{s_t}{\Psi_{t+1}} + \frac{1}{\theta}\left[\rho(1+r_{t+1})\right]^{-1/\theta}\cdot\frac{w_{t+1}(1+r_{t+1})^{-1}}{\Psi_{t+1}}$$

- - -

$$c_{2t+1}(1+r_{t+1})^{-1} = w_{t+1}(1+r_{t+1})^{-1} + s_t$$
$$c_{2t+1}(1+r_{t+1})^{-1} - s_t = w_{t+1}(1+r_{t+1})^{-1}$$

- - - -

$$\hookrightarrow \frac{\partial s_t}{\partial r_{t+1}} = \frac{1-\theta}{\theta}\left[\rho(1+r_{t+1})\right]^{-1/\theta}\frac{s_t}{\Psi_{t+1}} - \frac{1}{\theta}\left[\rho(1+r_{t+1})\right]^{-1/\theta}\frac{s_t}{\Psi_{t+1}} + \frac{1}{\theta}\left[\rho(1+r_{t+1})\right]^{-1/\theta}\frac{c_{2t+1}(1+r_{t+1})^{-1}}{\Psi_{t+1}}$$

$$= -\left[\rho(1+r_{t+1})\right]^{-1/\theta}\frac{s_t}{\Psi_{t+1}} + \frac{1}{\theta}\left[\rho(1+r_{t+1})\right]^{-1/\theta}\frac{c_{2t+1}(1+r_{t+1})^{-1}}{\Psi_{t+1}}$$

$$= \underbrace{\frac{\left[\rho(1+r_{t+1})\right]^{-1/\theta}}{\Psi_{t+1}}}_{(+)}\underbrace{\left(\frac{1}{\theta}c_{2t+1}(1+r_{t+1})^{-1} - s_t\right)}_{(?)}$$

(i) $\frac{\partial s_t}{\partial r_{t+1}} = 0$ if $\frac{1}{\theta}c_{2t+1}(1+r_{t+1})^{-1} - s_t = 0$

$$\frac{1}{\theta} = \frac{(1+r_{t+1})s_t}{c_{2t+1}} = 1 - w_{t+1}/c_{2t+1} = \frac{c_{2t+1}-w_{t+1}}{c_{2t+1}}$$

$$\theta = c_{2t+1}/c_{2t+1}-w_{t+1} = 1+\varepsilon \quad \text{for } \varepsilon > 0$$

(ii) $\frac{\partial s_t}{\partial r_{t+1}} > 0$ if $\frac{1}{\theta}c_{2t+1}(1+r_{t+1})^{-1} - s_t > 0$

$$\Rightarrow \theta < c_{2t+1}/c_{2t+1}-w_{t+1} = 1+\varepsilon \quad \text{for } \varepsilon > 0$$

(iii) $\frac{\partial s_t}{\partial r_{t+1}} < 0$ if $\frac{1}{\theta}c_{2t+1}(1+r_{t+1})^{-1} - s_t < 0$

$$\Rightarrow \theta > c_{2t+1}/c_{2t+1}-w_{t+1} = 1+\varepsilon \quad \text{for } \varepsilon > 0$$

Thus, working during second period of life increases the value of θ needed to make the income and substitution effects offsetting. Specifically, a higher level of risk aversion ($\theta > 1+\varepsilon$ for $\varepsilon > 0$) is needed for the income effect to dominate enough to where $\partial s_t/\partial r_{t+1} < 0$.

4. $$U_t = \frac{c_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{c_{2t+1}^{1-\theta}}{1-\theta} + \frac{1+n}{(1+\rho)(1+\phi)} U_{t+1} \qquad \rho \in (0,1) \qquad \phi \geq 0$$

$$r_t = f'(k_t) \qquad \delta = 0$$

$$w_t = f(k_t) - k_t f'(k_t) \qquad g = 0 \quad A_0 = 1$$

$$L_{t+1} = (1+n) L_t \qquad n \in (0,1)$$

a. $$U_t = u(c_{1t}) + \frac{1}{1+\rho} u(c_{2t+1}) + \frac{1+n}{(1+\rho)(1+\phi)} \left\{ u(c_{1t+1}) + \frac{1}{1+\rho} u(c_{2t+2}) + \frac{1+n}{(1+\rho)(1+\phi)} U_{t+2} \right\}$$

$$= \left\{ u(c_{1t}) + \frac{1}{1+\rho} u(c_{2t+1}) \right\} + \frac{1+n}{(1+\rho)(1+\phi)} \left\{ u(c_{1t+1}) + \frac{1}{1+\rho} u(c_{2t+2}) \right\} + \left(\frac{1+n}{(1+\rho)(1+\phi)} \right)^2 U_{t+2}$$

$$\vdots$$

$$= \sum_{i=0}^{\infty} \left(\frac{1+n}{(1+\rho)(1+\phi)} \right)^i \left[u(c_{1t+i}) + \frac{1}{1+\rho} u(c_{2t+1+i}) \right].$$

$$\lim_{T \to \infty} \left(\frac{1+n}{(1+\rho)(1+\phi)} \right)^T U_{t+T} = 0 \quad \text{if} \quad 1+n < (1+\rho)(1+\phi)$$

b. $\max_{c_{1t}, c_{2t+1}, s_t, b_{t+1}} \quad U_t \quad st \quad c_{1t} + s_t = w_t + b_t$

$$c_{2t+1} + (1+n) b_{t+1} = (1+r_{t+1}) s_t$$

$$b_{t+1} \geq 0 \quad \forall i$$

$$\begin{cases} c_{1t} = w_t + b_t - s_t \\ c_{2t+1} = (1+r_{t+1}) s_t - (1+n) b_{t+1} \\ c_{1t+1} = w_{t+1} + b_{t+1} - s_{t+1} \end{cases}$$

$$\max_{s_t, b_{t+1}} \left\{ U(c_{1t}) + \frac{1}{1+\rho} U(c_{2t+1}) \right\} + \frac{1+n}{(1+\rho)(1+\phi)} \left\{ U(c_{1t+1}) + \frac{1}{1+\rho} U(c_{2t+2}) \right\} + \ldots$$

$$s_t: \quad -U'(c_{1t}) + \frac{1}{1+\rho} U'(c_{2t+1})(1+r_{t+1}) = 0$$

$$b_{t+1}: \quad -\frac{1}{1+\rho} U'(c_{2t+1})(1+n) + \frac{1+n}{(1+\rho)(1+\phi)} U'(c_{1t+1}) = 0$$

c. $-c_{1t}^{-\theta} + \frac{1}{1+\rho} c_{2t+1}^{-\theta} (1+r_{t+1}) = 0$

$$\frac{c_{2t+1}}{c_{1t}} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta} \qquad \text{if } r_{t+1} > \rho, \quad c_{2t+1} > c_{1t}$$

$$-\frac{\cancel{1+n}}{\cancel{1+\rho}} c_{2t+1}^{-\theta} + \frac{\cancel{1+n}}{\cancel{(1+\rho)}(1+\phi)} c_{1t+1}^{-\theta} = 0$$

$$\frac{c_{1t+1}^{-\theta}}{c_{2t+1}^{-\theta}} = 1+\phi$$

$$\frac{c_{2t+1}}{c_{1t+1}} = (1+\phi)^{1/\theta} \implies \frac{c_{2t}}{c_{1t}} = (1+\phi)^{1/\theta}$$

$$\text{if } \phi > 0, \quad c_{2t} > c_{1t}$$

d. $c_t \equiv \dfrac{c_{1t}L_t + c_{2t}L_{t-1}}{L_t} = c_{1t} + c_{2t}(1+n)^{-1}$

$$\Rightarrow \quad c_{t+1}/c_t = \frac{c_{1t+1} + c_{2t+1}(1+n)^{-1}}{c_{1t} + c_{2t}(1+n)^{-1}}$$

express in terms of c_{1t}:

$$c_{2t} = c_{1t}(1+\phi)^{1/\theta}$$

$$c_{2t+1} = c_{1t}\left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta}$$

$$c_{1t+1} = c_{2t+1}(1+\phi)^{-1/\theta} = c_{1t}\left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta}(1+\phi)^{-1/\theta}$$

$$\hookrightarrow \quad \frac{c_{t+1}}{c_t} = \frac{\cancel{c_{1t}}\left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta}(1+\phi)^{-1/\theta} + \cancel{c_{1t}}\left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta}(1+n)^{-1}}{\cancel{c_{1t}} + \cancel{c_{1t}}(1+\phi)^{1/\theta}(1+n)^{-1}}$$

$$= \frac{\left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta}\left\{\cancel{(1+\phi)^{-1/\theta} + (1+n)^{-1}}\right\}}{(1+\phi)^{1/\theta}\left\{\cancel{(1+\phi)^{-1/\theta} + (1+n)^{-1}}\right\}}$$

$$= \left(\frac{1+r_{t+1}}{(1+\rho)(1+\phi)}\right)^{1/\theta}$$

if $1+r_{t+1} > (1+\rho)(1+\phi)$

$> 1+\rho+\phi+\rho\phi$

$r_{t+1} > \rho+\phi+\rho\phi$, then $c_{t+1} > c_t$