UTILIZING THE POWER OF TECHNOLOGY TO VISUALIZE MATHEMATICS

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ABSTRACT:

The purpose of this paper is to share some specific examples in which technology can empower students to visualize mathematics. These are examples from the literature which the author has utilized in instruction and are taken from the areas of precalculus, calculus, geometry, and probability. Placed in the context of research involving deep learning and visualization in mathematics, the examples utilize graphing calculators, dynamic geometry software, and applets on the web.

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The purpose of this paper and the accompanying conference presentation is to share some specific examples in which technology empowers students to visualize mathematics. These examples are not original to this presenter, but are examples from the literature which the author has utilized in instruction. The examples were chosen to span a variety of elementary mathematical topics as well as the utilization of various technologies. In order to place these examples in the context of research involving deep learning and visualization in mathematics, pertinent definitions and theories, highlights of related research, and consequent pedagogical implications follow.

DEEP LEARNING: Students who take a deep approach to learning as opposed to a surface approach are interested in the academic task, search for meaning in the task, personalize the task, integrate aspects of the task into a whole, see the relationships between this whole and previous knowledge, and try to theorize about the task (Joughin, 1992, Rhem, 1995).

VISUALIZATION: According to Arcavi (1999, p. 56) and based on earlier definitions by Zimmerman and Hershkowitz, "visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understanding." Using this definition, visualization has particular relevance to the aspects of deep learning which deal with integrating aspects of a task into a whole and seeing the relationships between this whole and previous knowledge.

RELATIONSHIP BETWEEN VISUALIZATION AND LEARNING MATHEMATICS: While visualization is very much a part of our biology and culture (Arcavi, 1999; Guzman 2002), paradoxically most students have a reluctance to visualize in mathematics (Eisenburg & Dreyfus, 1991). Duval (1999) asserts that only when individuals can go back and forth between various representations of mathematical concepts (for example, the visual and the analytic) does mathematical understanding occur. This is in accordance with the writings of Arcavi (1999). Ideally students would generate a "visual understanding which would include the ability to solve problems utilizing both visual and analytic thinking in concordance with one another, and …feel at ease in both domains" (Eisenburg and Dreyfus, 1991, p 29). However, creating and processing visual representations is difficult for most students (Eisenburg and Dreyfus, 1991).

Eisenburg and Dreyfus hypothesize three reasons why visual processing tends to be more difficult for students than analytic processing, bearing implications for the practice of mathematics teaching.

1. Beliefs: Teachers inadvertently convey the idea that visual approaches to mathematics are inferior to analytic approaches when they stress that a picture is not a proof.
2. Information Processing: Mathematicians in their own work tend to utilize visual methods of processing information, relying on diagrammatic representations in which relationships are preserved. When teaching, the same mathematicians tend to utilize analytic methods of processing information, relying on sequential representations of information. To explain this phenomenon, Eisenburg and Dreyfus call upon Chevallard's theory of mathematical didactics, which deals with the change knowledge undergoes as it is turned from scientific academic knowledge to knowledge taught in school. According to this theory, academic knowledge is intricate, containing many links and connections. For presentation purposes, this knowledge is typically linearized into a sequence, the end result being that the knowledge taught stresses computational procedure in order to allow for sequential presentation (Eisenburg and Dreyfus, 1991).
3. Cognitive efficiency. Being complex and concentrated collections of information, diagrams are utilitarian because they explicitly show important conceptual links between pieces of information, but they are not immediately intelligible because of this complexity; and it takes cognitive processing to make sense of them (Eisenburg and Dreyfus, 1991).

PEDAGOGICAL IMPLICATIONS: So what can instructors of mathematics do to encourage students in visualizing mathematics? In general both curricula and pedagogy need to be rethought, both in light of research concerning how individuals learn and in light of tools, including technological tools (Smith, 2002). Specific suggestions include:

- Giving students focused and specific experiences in interpreting diagrams/visuals of various types, in using them in solving problems, and in constructing useful diagrams themselves (Eisenburg and Dreyfus, 1991).
- Engaging students in active learning in which they construct mathematical knowledge in much the way a mathematician does. This involves transitioning the instructor's role from *sage on the stage* in which information is exclusively delivered by lecture to that of a *conductor orchestrating a symphony* in which the student takes a more active role in learning (Smith 2002; Rhem 1995).
- Utilizing technological tools now available for visualization in mathematics. However, the mere use of a technological tool does not guarantee enhanced learning, rather the appropriate use of technology in context of rethought curricula and pedagogy (Smith, 2002; Goldenburg, 1991, Tall, 2000).

SPECIFIC EXAMPLES OF UTILITY OF TECHNOLOGY IN VISUALIZING MATHEMATICS:

1. Using graphing technology to illustrate differentiability visually as local linearity (Tall, 2000; Tall 1990). Arcavi proposes that a visual solution to a problem can engage students with "meanings which can be easily bypassed by the symbolic solution of the problem" (1999, p. 62) and can bring geometry to the aid of what seems to be purely algebraic/symbolic processes (1999). To illustrate, one can symbolically show that a quadratic function is differentiable at the vertex, but an absolute value function is not. However, the visual process of repeated zooming in on each vertex allows us to associate a more geometric representation to the concept of differentiability/non-differentiability and the associated existence/non-existence of a tangent line at a particular point. After repeated zooming, the visual evidence seen in the TI-83 graphs in Table I that a tangent line does not exist at the vertex of the absolute value function while a horizontal tangent line exists at the

vertex of a quadratic function is compelling. Douglas Arnold has posted on the web a collection of animated gifs produced by Mathematica to illustrate a variety of calculus concepts graphically. These include a video of the visual effect of zooming on a graph.

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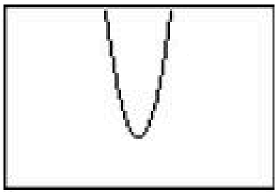
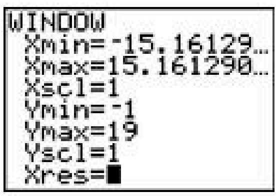
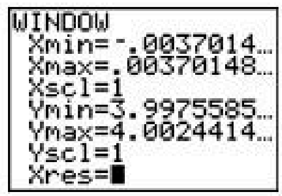
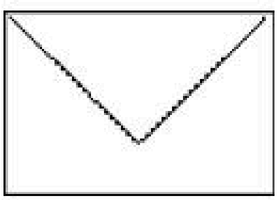
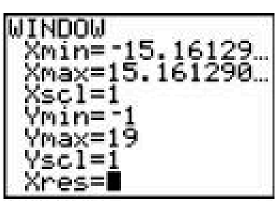
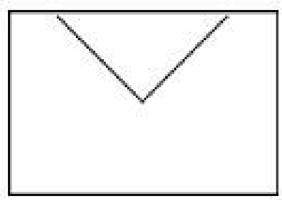
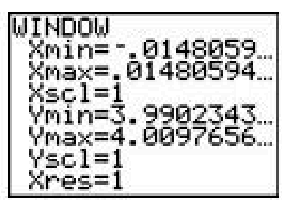
Table 1
(Differentiability as Local Linearity)

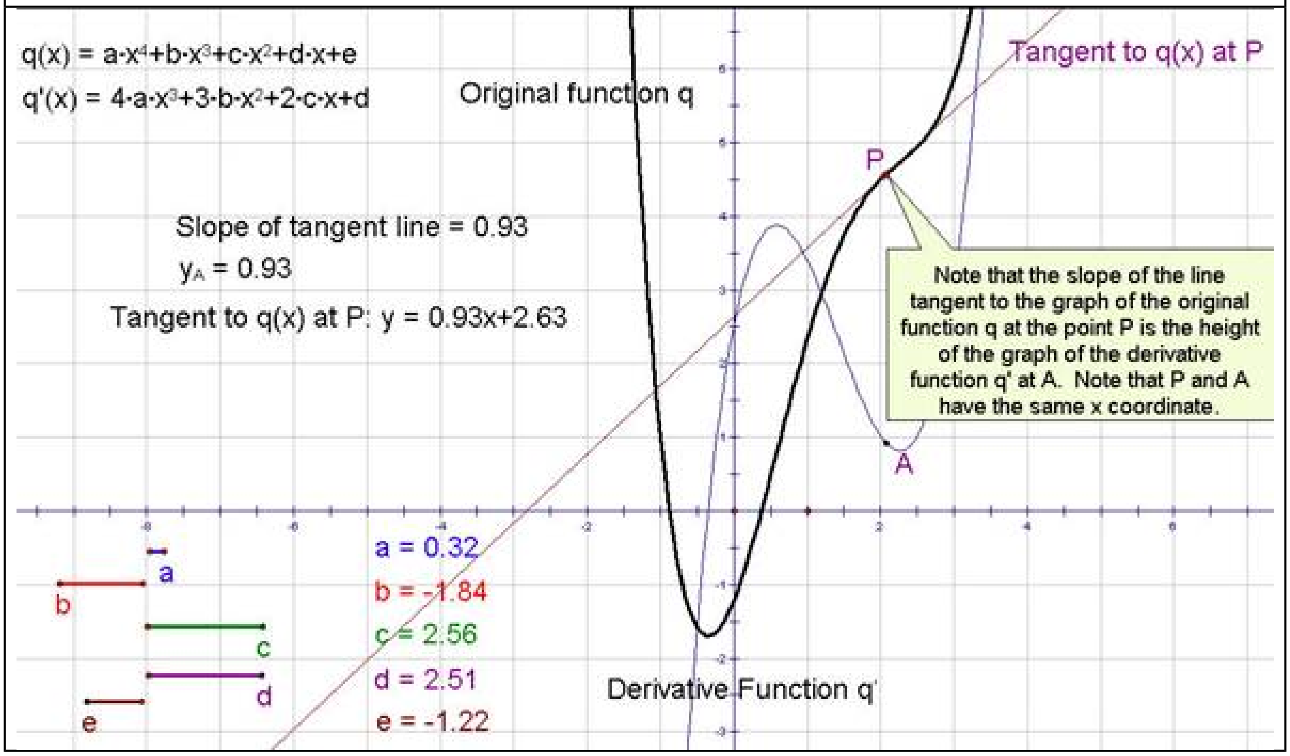
Graph of $y = x^2 + 4$ (with axes hidden)   	→	Graph of $y = x^2 + 4$ after repeated zooming at the vertex (0,4). (with axes hidden)  
Graph of $y = \|x\| + 4$ (with axes hidden)   	→	Graph of $y = \|x\| + 4$ after repeated zooming at the vertex (0,4) (with axes hidden)  

2. Using graphing technology to show the relationship between the graph of a function and its derivative as well as to show the relationship between the position of a tangent line and concavity at a point (Zimmerman, 1990; Tall, 1990). The benefits of using visualization in this context are those benefits described by Arcavi (1999) in the previous example. Produced with Geometer's Sketchpad, Table II shows the graph of a quartic function and its derivative. It features the tangent line at a particular point which can be animated along the graph, and explicitly includes the slope and equation of the tangent line and the height of the derivative graph at the point of tangency. A grid was used

to aid students in estimation of slopes. When the point of tangency is animated, the fact that the tangent line is above the graph at points where the graph is concave down and below at points where the graph is concave up can be visually ascertained. In addition, students can manipulate sliders in order to vary the coefficients in the quartic function and thus receive immediate feedback regarding the effects of the various coefficients on the graphs (Kutzler, 2003). Alternatively, this interactive visualization module can also be produced easily using TI-Interactive.

Table II

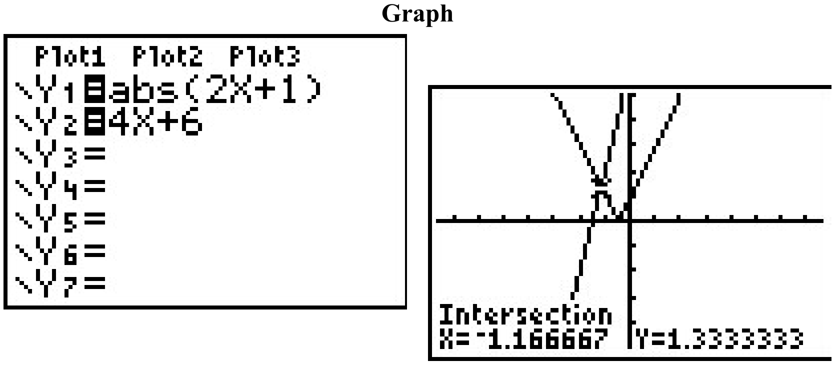
Relationship Between the Graph of a Function and its Derivative



3. Using graphing technology to produce a two-dimensional graph to analyze in order to solve an inequality in one variable (Rockswold, 2002). Arcavi proposes that a visual solution to a problem can provide a "possible way of resolving conflict between (correct) symbolic solutions and (incorrect) intuitions" (1999, p. 63). From analysis of the graphs in Table III, one can see that the solution to the inequality $|2x+1| < 4x+6$ is $x > \frac{-7}{6}$. This author has found that the algebraic solution of this inequality is problematic for most college algebra students and is often initially missed by prospective secondary school teachers as they approach their student teaching semester.

Table III

Solution of an Inequality in One Variable by Analyzing aTwo-Dimensional Graph



4. Technology allows students to visualize the dynamics of mathematical processes (Cunningham, 1991), including those of random phenomenon to order to better understand the conceptual underpinnings of probability and statistics (Weissglass & Cummings, 1991). Because individuals do not typically have adequate experience with random phenomena, their intuition concerning randomness is often faulty (Moore, 1990). Mathlets on the web such as those written by Siegrist (www.math.uah.edu/stat) allow for the simulation of random phenomena. The spinner applet, for example, can be used to dynamically visualize stability of frequencies, often called the law of large numbers. As the number of trials of an experiment increases without bound, the relative frequencies of the outcomes approach the theoretical probabilities of the outcomes (Moore, 2000). When running the applet the visual effect of the large change in the heights of the bars on the relative frequency histogram while the number of spins is small, compared to the small change when the number of spins is large, is impressive. This provides a compelling visualization of stability of frequencies. The spinner probability simulation on the TI-73 graphing calculator allows for the same visualization.
5. Dynamic geometry software packages such as the Geometer's Sketchpad and Cabri give students a powerful visual tool for gathering geometric data in order to reason inductively and formulate conjectures, the same process which mathematicians use in their mathematical research. The software helps students move through the van Hiele levels of Visualization, Analysis, and Informal Deduction and can be used to motivate students to achieve the levels of Formal Deduction and Rigor. Specifically dynamic geometry software allows one to construct geometric figures using tools of the software package. However, unlike the static images constructed by hand using straightedge and compass, the Sketchpad figures can be manipulated, having the variant properties changed by dragging. One can consequently observe invariants, producing large amounts of data to analyze (Schattschneider D. & King J, 1997). Quesda and Wantanabe (1996) have sited examples of original results secondary school students have created, aided by dynamic geometry packages.

As was mentioned previously, not only are the Geometer's Sketchpad and other dynamic geometry software packages useful for engaging students in active learning in geometry, in that it gives them a tool which facilitates their engagement in the same type of behavior which mathematicians engage as they do mathematical research, but it allows students to undergo similar experiences with functions because of its graphical capabilities. Schattschneider and King (1997) quote Morrow as saying that Sketchpad has particular potential with regard to enhancing visualization in that 'the capacity for students to make instantaneous and precise variations to their own visual representations adds a new dynamic dimension whose implications are only beginning to be understood.' By using Sketchpad as an aid in visualization, students find meaning in concepts such as variable, function, and invariance.

PITFALLS/CONSIDERATIONS FOR THE CLASSROOM WITH REGARD TO VISUALIZATION: Students do not necessarily perceive what teachers and researchers see in visual representations. Students unfamiliar with underlying concepts see "irrelevancies" which are dismissed by the expert's vision. For example, students often focus on jaggedness in graphs produced as a result of limitations of the calculator screen resolution or place significance on the fact that a computer graphics package or graphing calculator graphs from left to right. If a non-complete graph results, some students automatically assume that the missing part must be on the right because the graphing device produced the graph from left to right. Instructors need to be aware of this, assess what students perceive in visual representations (Ward; Arcavi, 1999), and as Eisenburg and Dreyfus (1991) suggest, give students focused and specific experiences in interpreting visual representations, in using visual representations in solving problems, and in constructing useful visual representations themselves.

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Virtual Laboratories in Probability/Statistics. Siegrist, Kyle. www.math.uah.edu/stat

Dynamic Geometry Software:
Cabri. Texas Instruments. www.ticalc.com
Geometer's Sketchpad. Key Curriculum Press. www.keypress.com

Graphing Calculators:
TI-73 Graphing Calculator. Texas Instruments. www.ticalc.com
TI-83 Graphing Calculator. Texas Instruments. www.ticalc.com

SOFTWARE NOT CITED IN PAPER BUT USEFUL FOR VISUALIZATION

Computer Algebra System/Graphics Package/Mathematical Word Processor:
Scientific Notebook http://www.mackichan.com

Video Production Software to Make Movies of Actions on the Computer:
Hypercam http://www.hyperionics.com/hc/index.asp
Camtasia. www.techsmith.com

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