Discrete Choice Dynamic Programming Models: Search, Matching, & Generalized Models

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Contents

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1 Introduction to Discrete Choice Dynamic Programming (DCDP) Models

Until recently, the economic analysis of discrete decisions has been an underrepresented field in economics. Because traditional analytical tools – such as derivatives – are not applicable in discrete choices, computational techniques form a barrier to entry into the field. With the increasing use of computational tools in economics, however, DCDP models have increased in popularity to where most fields have active research agendas. See Keane and Wolpin (2008) for a review.

Thus, the purpose of these notes is to give the reader some minimal knowledge on the computational tools necessary for the analysis – both quantitative and empirical – of DCDP models. The three main conduits for this study are McCall's job search model, Jovanovic's matching model, and generalized DCDP models (more realistic versions of the previous two models).

2 Search: McCall's Model

J.J. McCall's model[1] is of an unemployed worker searching for a job. Each period the worker gets offered a job (at time t we would say w_t). If the job offer is accepted, she stays at the job forever and gives up unemployment compensation c. Obviously, if she rejects her period t offer, she will earn c units of unemployment insurance and wait till next period $(t+1)$ for a new offer. If her utility is linear, then an accepted time t job offer gives her momentary utility of $u_t = w_t$; formally we define utility as $u(w_t) = w_t$. If she rejects the job, her momentary utility is $u_t = c$ or $u(c) = c$.

What role does the unemployment insurance play? Most likely the higher the c the

[1] See McCall (1970), "Economics of information and job search," *Quarterly Journal of Economics* **84**, pp. 113-126.

pickier the worker will be. That is, higher unemployment compensation leads to higher unemployment. Also, note that the current value of a worker depends on the wage offer; if the wage is accepted then utility is w_t and if rejected utility is c. Therefore, utility is ultimately determined by w_t; we say that the value of being an agent who has been offered w_t at time t is $V(w_t)$. Finally, we will take as given that search and matching models have common solution forms. A solution is denoted a reservation wage $\bar{w}$ and satisfies the following conditions: if $w_t > \bar{w}$ then accept the offer and if $w_t \leq \bar{w}$ then accept the offer.

2.1 The Setup at time t

The tension in the model is that the agent doesn't know, at time t, what future job offers lay ahead. Therefore, if the agent accepts the wage offer at time t then the agent foregoes any future possible windfall from high job offers. Alternatively, if the agent rejects the offer she then looses out of a sure stream of income.

Using economic reasoning, the time t value of accepting the job is: $V(w_t) = w_t + \beta V(w_t)$. Or, rearranging gives:

$$V(w_t) = w_t/(1-\beta).$$

Alternatively, if the agent rejects the job offer at time t then she collects unemployment c and goes on the job market next period. Her discounted expectation of next periods return is therefore: $\beta E_t\left[V(w_{t+1})\right]$. Here we have used the conditional expectation. The value of rejecting the time t wage is thus:

$$V(w_t) = c + \beta E_t\left[V(w_{t+1})\right].$$

Evidently, the Bellman's equation is:

$$V(w_t) = \max_{a_t \in \{\text{accept},\text{reject}\}} \left\{w_t/(1-\beta), c + \beta E_t\left[V(w_{t+1})\right]\right\},$$

where a_t is the optimal solution and is either $a_t = 1$ if she accepts the wage offer or is $a_t = 2$ if she rejects the wage offer.

2.2 Statistical Preliminaries

Let the random variable $x_t \in [0, 1, \ldots, K]$ follow a Markov chain with a transition probability matrix of:

$$\Pr\{x_{t+1}|x_t\} = \mathbf{P}$$

where $\Pr\{x_{t+1} = K|x_t = J\} = \mathbf{P}_{J,K}$. Note that $\mathbf{P}$ defines a conditional distribution for x. The unconditional distribution – or limiting distribution – for x can be computed by normalizing the absolute value of the eigenvector associated with the unit eigenvalue for $\mathbf{P}'$. In matlab, the code is: `[v,d]=eig(P'); mu = abs(v(:,1))./sum(abs(v(:,1)))` when the first eigenvalue is of value one. The unconditional distribution is denoted as:

$$\Pr\{x_t\} = \boldsymbol{\mu}$$

where $\Pr\{x_t = J\} = \boldsymbol{\mu}_J$.

Question 1: *Using matlab, find the limiting distribution of a random variable with transition probability matrix of:*

$$\mathbf{P} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

Question 2: *Using matlab, find the limiting distribution of a random variable with transition probability matrix of:*

$$\mathbf{P} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 1/3 & 2/3 & 0 \end{bmatrix}.$$

The mean of $\mathbf{x}_t$ is defined as

$$E[x_t] = \boldsymbol{\mu}'\mathbf{x}$$

where $\mathbf{x} =[0, 1, \ldots, K]'$. For example, suppose that $\mathbf{x} =[0, 1, 2, 3]'$ and $\boldsymbol{\mu} = [0.25, 0.25, 0.30, 0.20]'$, Then,

$$E[x_t] = \boldsymbol{\mu}'\mathbf{x} = [0.25, 0.25, 0.30, 0.20] \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = 1.45$$

The conditional mean of the random variable given information at time t, say x_t, is defined as:

$$E[x_{t+1}|x_t = J] = \sum_{j=1}^{K} \mathbf{P}_{J,j}\mathbf{x}_{j,1}.$$

In matrix form its just the product $\mathbf{P} \cdot \mathbf{x}$.

Later in these notes we will use a trick of the expectation operator that states, for any $\bar{X}$:

$$E[x_t] = \Pr\left\{x_t < \bar{X}\right\} E\left\{x_t|x_t < \bar{X}\right\} + \Pr\left\{x_t \geq \bar{X}\right\} E\left\{x_t|x_t \geq \bar{X}\right\}.$$

The same trick can be used on functions defined on random variables, say $G(x_t)$:

$$E[G(x_t)] = \Pr\left\{x_t < \bar{X}\right\} E\left\{G(x_t)|x_t < \bar{X}\right\} + \Pr\left\{x_t \geq \bar{X}\right\} E\left\{G(x_t)|x_t \geq \bar{X}\right\}$$

For example, when $\bar{X} = 2$ we see that $\Pr\left\{x_t < \bar{X}\right\} = 0.25+0.25 = 0.50$ and $\Pr\left\{x_t \geq \bar{X}\right\} = 0.30+0.20 = 0.50$. And, $\Pr\left\{x_t = 0|x_t < \bar{X}\right\} = 0.25/(0.25+0.25) = 0.50$, $\Pr\left\{x_t = 1|x_t < \bar{X}\right\} = 0.25/(0.25+0.25)$, $\Pr\left\{x_t = 2|x_t \geq \bar{X}\right\} = 0.30/(0.30+0.20) = 0.60$, and $\Pr\left\{x_t = 3|x_t \geq \bar{X}\right\} = 0.20/(0.30+0.20) = 0.40$. Therefore $E\left\{x_t|x_t < \bar{X}\right\} = 0.5{\cdot}0+0.5{\cdot}1 = 0.5$ and $E\left\{x_t|x_t \geq \bar{X}\right\} =$

$0.6 \cdot 2 + 0.4 \cdot 3 = 2.4$. Finally,

$$\begin{aligned} E[x_t] &= \Pr\left\{x_t < \bar{X}\right\} E\left\{x_t | x_t < \bar{X}\right\} + \Pr\left\{x_t \geq \bar{X}\right\} E\left\{x_t | x_t \geq \bar{X}\right\} \\ &= 0.50 \cdot 0.50 + 0.5 \cdot 2.4 \\ &= 1.45 \end{aligned}$$

2.3 Analytical Solution

How can we solve for $\bar{w}$? Solving for the reservation wage analytically is very difficult. However, it is possible to solve for $\bar{w}$ if we make some simple assumptions on the distribution of wages. Specifically, let the wage offers be distributed discretely $w = [0, 1, \ldots, 100]'$ and by the following 101×101 transition probability matrix:

$$\mathbf{P} = \begin{bmatrix} 1/101 & 1/101 & \cdots & 1/101 \\ 1/101 & 1/101 & \cdots & 1/101 \\ \vdots & \vdots & \ddots & \vdots \\ 1/101 & 1/101 & \cdots & 1/101 \end{bmatrix}$$

that leads to a limiting distribution of $\boldsymbol{\mu} = [1/101, 1/101, \ldots, 1/101]'$.

Question 3: *Show, using matlab, that the limiting distribution of the transition probability matrix is indeed* $\mu = [1/101, 1/101, \ldots, 1/101]'$.

It turns out that with our assumptions about the distribution that the unconditional expectation of future wages are the same as the conditional expectation of the wage: $E[w_{t+1}] = E_t[w_{t+1}] = 50$. Additionally, the unconditional expectation of functions defined on future wages, such as the value function, are the same as the conditional expectation of the same function: $E[V(w_{t+1})] = E_t[V(w_{t+1})]$.

Question 4: *Make a convincing argument for why the two expectations operators, $\{E[\cdot], E_t[\cdot]\}$, are the same with the assumptions that we have placed on the distribution of w_t.*

If the consumer is offered a wage at the reservation, then she should be indifferent between working or staying unemployed. That is:

$$\frac{\bar{w}}{1-\beta} = c + \beta A \tag{1}$$

where we have used defined $E_t[V(w_{t+1})] = A$. We can further break A down into more specific parts using our trick of expectations and $E[\cdot] = E_t[\cdot]$:

$$A = E[V(w_{t+1})] = \left\{ \begin{array}{c} \Pr\{w_{t+1} < \bar{w}\} E\{V(w_{t+1})|w_{t+1} < \bar{w}\} + \\ \Pr\{w_{t+1} \geq \bar{w}\} E\{V(w_{t+1})|w_{t+1} \geq \bar{w}\} \end{array} \right\}. \tag{2}$$

Note that

$$\begin{aligned} E\{V(w_{t+1})|w_{t+1} < \bar{w}\} &= c + \beta E[V(w_{t+2})] \\ &= c + \beta A \end{aligned}$$

and

$$E\{V(w_{t+1})|w_{t+1} \geq \bar{w}\} = E\{w_{t+1}/(1-\beta)|w_{t+1} \geq \bar{w}\}.$$

Letting $\Pr\{w_{t+1} < \bar{w}\} = \pi(\bar{w})$ and $E\{w_{t+1}/(1-\beta)|w_{t+1} \geq \bar{w}\} = \mu(\bar{w})/(1-\beta)$ gives (2) as:

$$A = \pi(\bar{w})[c + \beta A] + [1 - \pi(\bar{w})]\frac{\mu(\bar{w})}{1-\beta}$$

Solving for A leads us to:

$$A = \pi(\bar{w})\frac{c}{[1-\beta\pi(\bar{w})]} + \frac{[1-\pi(\bar{w})]}{[1-\beta\pi(\bar{w})]}\frac{\mu(\bar{w})}{1-\beta}$$

substituting into (1) gives:

$$\frac{\bar{w}}{1-\beta} = c + \beta \left[\pi(\bar{w}) \frac{c}{[1-\beta\pi(\bar{w})]} + \frac{[1-\pi(\bar{w})]}{[1-\beta\pi(\bar{w})]} \frac{\mu(\bar{w})}{1-\beta} \right]$$

Question 5: *Show that, under our assumptions,* $\pi(\bar{w}) = \frac{\bar{w}}{101}$.

Question 6: *Show that, under our assumptions,* $\Pr\{w_{t+1} = \bar{w} + j | w_{t+1} \geq \bar{w}\} = \frac{\frac{1}{101}}{1-\frac{\bar{w}}{101}} = \frac{1}{101-\bar{w}}$, *for all positive* j.

Question 7: *Show that, under our assumptions,* $\mu(\bar{w}) = \bar{w}\frac{1}{101-\bar{w}} + (\bar{w}+1)\frac{1}{101-\bar{w}} + (\bar{w}+2)\frac{1}{101-\bar{w}} + \ldots + (\bar{w}+100-\bar{w})\frac{1}{101-\bar{w}}$.

Using the results above and from calculus $\mu(\bar{w}) \approx \frac{1}{2}\frac{101^2-\bar{w}^2}{101-\bar{w}}$ gives:

$$\frac{\bar{w}}{1-\beta} = c + \beta \left[\frac{\bar{w}}{101} \frac{c}{\left[1-\frac{\bar{w}}{101}\right]} + \frac{\left[1-\frac{\bar{w}}{101}\right]}{\left[1-\beta\frac{\bar{w}}{101}\right]} \frac{\frac{1}{2}\frac{101^2-\bar{w}^2}{101-\bar{w}}}{1-\beta} \right]$$

Letting $c = 10$ and $\beta = 0.99$ allows us to solve for $\bar{w} = 88.4$.

Question 8: *What effect do you think expanding the set of possible wages, say from* $w = [0, 1, \ldots, 100]'$ *to* $w = [0, 1, \ldots, 200]'$, *would have on the reservation wage?*

2.4 Quantitative Solution

Value function iteration can always be used to solve a Bellman's equation. Value function iteration would first guess $E[V(w_{t+1})] = 0$ for each w and find $V(w_t)^{<1>}$. This would be updated and used again to find a new value function denoted $V(w_t)^{<2>}$. In matrix form, we implement the guess and solve:

$$V(\mathbf{w})^{<1>} = \max_{a_t \in \{\text{accept,reject}\}} \left\{ \frac{1}{1-\beta}\mathbf{w}, \mathbf{c} \right\}$$

where $\mathbf{w} = [0, 1, \ldots, 100]'$ and $\mathbf{c} = [c, c, \ldots, c]'$. Note that the value function will be a 101×1 matrix defined by: $V(\mathbf{w})^{<1>} = [V(w_1)^{<1>}, V(w_2)^{<1>}, \ldots, V(w_{101})^{<1>}]'$. The new value function serves as a new guess (presumably closer to the solution due to the fact that the Bellman's forms a contraction) and is used to solve:

$$V(\mathbf{w})^{<2>} = \max_{a_t \in \{\text{accept,reject}\}} \left\{ \frac{1}{1-\beta}\mathbf{w}, \mathbf{c} + \beta \mathbf{P} V(\mathbf{w})^{<1>} \right\}.$$

A solution is found at iteration I when the statistic

$$\frac{\left\| V(\mathbf{w})^{<I>} - V(\mathbf{w})^{<I-1>} \right\|}{(1 + \| V(\mathbf{w})^{<I-1>} \|)}$$

is reasonably small (where $\|\mathbf{x}\| = \sqrt{\mathbf{x}'\mathbf{x}}$). Note that a solution for this problem will be a reservation wage $\bar{w}$.

Consider the following `matlab` code below. Here we have used the dynamic programing algorithm presented above. As you can see, in the initial stage the worker always rejects wage offers below 88, $\bar{w} = 88$.

```
%
% mccall1.m is an example matlab program
% to solve a simple maacall searcgmodel
%

% ----- save output to file ----- %

clear all
diary mccall1.mout
diary off
delete mccall1.mout
diary mccall1.mout

% ----- display date and time of computation ----- %

%format short
date
time0 = clock;

% ----- set control vars ----- %

iters  = 200;
states = 101;

% ------ set parameters ------ %
```

```
beta = 0.99;
c    = 10*ones(states,1);
P    = 1/states*ones(states,states);

% ----- find limiting distribution ----- %

[v,d]=eig(P') ;

% - find the row of the unit eigenvalue - %

[d1,d2]=find(fix(diag(d)+eps))

% - normalize eigenvector - %

mu = abs(v(:,d1))./sum(abs(v(:,d1))) ;

% ----- discretize wage ----- %

w = zeros(states,1);

for j = 1:(states-1);
 w(j+1,1) = w(j,1) + 1;
end;

% ----- value function iteration ----- %

% - initial guess of value function - %

v0  = zeros(states,1);

% - start iterations - %

crit1 = 3;

for j = 1:iters ;
 if crit1 < 0.0001, break, end;

% - compute second period payoffs and policy - %

  payoff11 = 1/(1-beta)*w ;
  payoff12 = c + beta*P*v0 ;
  payoff0  = [payoff11  payoff12 ];

  [newv0,policy0] = max(payoff0') ;

% - put matrices back to correct dimension - %

  newv0   = newv0';
  policy0 = policy0';

 iter = j;

% - compute criterion - %

 crit1 = sqrt((newv0-v0)'*(newv0-v0))/(1+sqrt(v0'*v0)) ;

% - update - %

 v0 = newv0;

end;

% ----- print solutions ----- %

```

```
[w,v0,policy0],  [crit1, iter]

w0bar = sum(any(policy0-1,2))

comptime = etime(clock, time0)
diary off
```

```

ans =

03-Sep-2008


d1 =

   101


d2 =

     1


ans =

  1.0e+003 *

         0    8.7831    0.0020
    0.0010    8.7831    0.0020
    0.0020    8.7831    0.0020
    0.0030    8.7831    0.0020
    0.0040    8.7831    0.0020
    0.0050    8.7831    0.0020
    0.0060    8.7831    0.0020
    0.0070    8.7831    0.0020
    0.0080    8.7831    0.0020
    0.0090    8.7831    0.0020
    0.0100    8.7831    0.0020
    0.0110    8.7831    0.0020
    0.0120    8.7831    0.0020
    0.0130    8.7831    0.0020
    0.0140    8.7831    0.0020
    0.0150    8.7831    0.0020
    0.0160    8.7831    0.0020
    0.0170    8.7831    0.0020
    0.0180    8.7831    0.0020
    0.0190    8.7831    0.0020
    0.0200    8.7831    0.0020
    0.0210    8.7831    0.0020
    0.0220    8.7831    0.0020
    0.0230    8.7831    0.0020
    0.0240    8.7831    0.0020
    0.0250    8.7831    0.0020
    0.0260    8.7831    0.0020
    0.0270    8.7831    0.0020
    0.0280    8.7831    0.0020
    0.0290    8.7831    0.0020
```

51	0.0300	8.7831	0.0020
52	0.0310	8.7831	0.0020
53	0.0320	8.7831	0.0020
54	0.0330	8.7831	0.0020
55	0.0340	8.7831	0.0020
56	0.0350	8.7831	0.0020
57	0.0360	8.7831	0.0020
58	0.0370	8.7831	0.0020
59	0.0380	8.7831	0.0020
60	0.0390	8.7831	0.0020
61	0.0400	8.7831	0.0020
62	0.0410	8.7831	0.0020
63	0.0420	8.7831	0.0020
64	0.0430	8.7831	0.0020
65	0.0440	8.7831	0.0020
66	0.0450	8.7831	0.0020
67	0.0460	8.7831	0.0020
68	0.0470	8.7831	0.0020
69	0.0480	8.7831	0.0020
70	0.0490	8.7831	0.0020
71	0.0500	8.7831	0.0020
72	0.0510	8.7831	0.0020
73	0.0520	8.7831	0.0020
74	0.0530	8.7831	0.0020
75	0.0540	8.7831	0.0020
76	0.0550	8.7831	0.0020
77	0.0560	8.7831	0.0020
78	0.0570	8.7831	0.0020
79	0.0580	8.7831	0.0020
80	0.0590	8.7831	0.0020
81	0.0600	8.7831	0.0020
82	0.0610	8.7831	0.0020
83	0.0620	8.7831	0.0020
84	0.0630	8.7831	0.0020
85	0.0640	8.7831	0.0020
86	0.0650	8.7831	0.0020
87	0.0660	8.7831	0.0020
88	0.0670	8.7831	0.0020
89	0.0680	8.7831	0.0020
90	0.0690	8.7831	0.0020
91	0.0700	8.7831	0.0020
92	0.0710	8.7831	0.0020
93	0.0720	8.7831	0.0020
94	0.0730	8.7831	0.0020
95	0.0740	8.7831	0.0020
96	0.0750	8.7831	0.0020
97	0.0760	8.7831	0.0020
98	0.0770	8.7831	0.0020
99	0.0780	8.7831	0.0020
100	0.0790	8.7831	0.0020
101	0.0800	8.7831	0.0020
102	0.0810	8.7831	0.0020
103	0.0820	8.7831	0.0020
104	0.0830	8.7831	0.0020
105	0.0840	8.7831	0.0020
106	0.0850	8.7831	0.0020
107	0.0860	8.7831	0.0020
108	0.0870	8.7831	0.0020
109	0.0880	8.8000	0.0010
110	0.0890	8.9000	0.0010
111	0.0900	9.0000	0.0010
112	0.0910	9.1000	0.0010
113	0.0920	9.2000	0.0010
114	0.0930	9.3000	0.0010
115	0.0940	9.4000	0.0010

```
      0.0950    9.5000    0.0010
      0.0960    9.6000    0.0010
      0.0970    9.7000    0.0010
      0.0980    9.8000    0.0010
      0.0990    9.9000    0.0010
      0.1000   10.0000    0.0010


ans =

      0.0001   40.0000


w0bar =

      88


comptime =

      0.1250

```

Question 9: *Alter the code to compute the solution where* $w = [0, 1, \ldots, 200]'$.

3 Matching: A Simple Jovanovic Model

At time t, the agent is offered a wage w_t with unconditional probability $\Pr\{w_t = w_i\} = \mu_i$ where $w_i \in \{0, 1, 2, 3, \ldots, 9\}$ (note that a wage of zero is an unemployed person). The agent may reject the offer and obtain unemployment insurance of c and a chance at a new offer next period. If the agent accepts the job, she will work for w_t today. The value of being an agent who has been offered w_t at time t is denoted $V_0(w_t)$.

If the agent accepts the job, next period $(t+1)$ the productivity of the **match** will be realized and the firm will offer a new wage w_{t+1}. The worker may accept or reject the new wage. If the worker accepts the w_{t+1} wage, she will work at it forever. The value of being a worker who has been offered w_{t+1} at time $t+1$ is denoted $V_1(w_{t+1})$.

Notice the subscripts in $V_0(w_t)$ and $V_1(w_{t+1})$. They represent the value function for before and after the wage match has been realized. For example, $V_1(w_t)$ is the value function of our agent who has realized and been offered her true value to the company, w_t, at time t.

3.1 The Setup at time t

The tension in the model is that the agent doesn't know, at time t, how the match will be realized. Therefore, if the agent accepts the wage offer at time t then the agent has a discounted expectation of her match: $\beta E_t \left[V_1(w_{t+1})\right]$ where E_t denotes the conditional expectation with respect to all information at time t. The time t return of accepting the job is therefore $w_t + \beta E_t \left[V_1(w_{t+1})\right]$.

Alternatively, if the agent rejects the job offer at time t then she collects unemployment c and goes on the job market next period. Her discounted expectation of next periods return is therefore: $\beta E \left[V_0(w_{t+1})\right]$. Here we have used the unconditional expectation.

The agent's time t problem is therefore:

$$V_0(w_t) = \max_{a_t \in \{\text{accept,reject}\}} \left\{ w_t + \beta E_t \left[V_1(w_{t+1})\right], c + \beta E \left[V_0(w_{t+1})\right] \right\},$$

where a_t is the optimal solution and is either $a_t = 1$ if she accepts the wage offer or is $a_t = 2$ if she rejects the wage offer,

3.2 The setup at time $t+1$

If the agent has accepted the wage offer in the previous period, the time $t+1$ one problem is to accept the wage offer for ever w_{t+1} or go on the job market and search for a new job. The agent's time $t+1$ problem is therefore:

$$V_1(w_{t+1}) = \max_{a_t \in \{\text{accept,reject}\}} \left\{ w_{t+1} + \beta V_1(w_{t+1}), c + \beta E \left[V_0(w_{t+2})\right] \right\}.$$

Or:

$$V_1(w_{t+1}) = \max_{a_t \in \{\text{accept,reject}\}} \left\{ \frac{w_{t+1}}{1-\beta}, c + \beta E \left[V_0(w_{t+2})\right] \right\}.$$

3.3 Distributional assumptions

To quantitatively solve the model we can make assumptions on the distributions of the wage offers and the wage matches. As stated before, a wage offer w_t with unconditional probability $\Pr(w_t = w_i) = \mu_i$. In matrix form, we write $\boldsymbol{\mu} = [\mu_1, \mu_2, \ldots, \mu_{10}]'$. Alternatively, assume that the match evolves according to a markov chain with transition probability matrix defined by $P_{j,i} = \Pr(w_{t+1} = w_i | w_t = w_j)$. Further, we impose how the match evolves by zero restrictions represented in matrix form for the transition probability matrix:

$$\mathbf{P} = \begin{bmatrix} P_{1,1} & P_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 \\ P_{2,1} & P_{2,2} & P_{2,3} & 0 & 0 & 0 & 0 & 0 \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} & 0 & 0 & 0 & 0 \\ P_{4,1} & 0 & P_{4,3} & P_{4,4} & P_{4,5} & 0 & 0 & 0 \\ P_{5,1} & 0 & 0 & P_{5,4} & P_{5,5} & P_{5,6} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ P_{9,1} & 0 & 0 & 0 & 0 & P_{9,8} & P_{9,9} & P_{9,10} \\ P_{10,1} & 0 & 0 & 0 & 0 & 0 & P_{10,9} & P_{10,10} \end{bmatrix}$$

The first state is where the worker earns a zero wage. This could be where she interns for free. Or, more realistically, where the net benefit (net of costs associated with work like transportation to and from work) are zero.

3.4 Solving the model: value function iteration

Notice that the two Bellmans equations can be combined into one to completely eliminate V_1:

$$V_0(w_t) = \max_{a_t \in \{\text{accept,reject}\}} \left\{ \begin{array}{l} w_t + \beta E_t \left[\max_{a_{t+1} \in \{\text{accept,reject}\}} \left\{ \frac{w_{t+1}}{1-\beta}, c + \beta E \left[V_0(w_{t+2}) \right] \right\} \right], \\ c + \beta E \left[V_0(w_{t+1}) \right] \end{array} \right\}$$

Value function iteration would first guess $E[V_0(w_{t+2})] = E[V_0(w_{t+1})] = 0$ for each w and find $V_0(w_t)^{<1>}$. This would be updated and used again to find a new value function denoted $V_0(w_t)^{<2>}$.

In matrix form, we implement the guess and solve:

$$V_0(\mathbf{w})^{<1>} = \max_{a_t \in \{\text{accept,reject}\}} \left\{ \mathbf{w} + \beta \mathbf{P} \left[\max_{a_{t+1} \in \{\text{accept,reject}\}} \left\{ \frac{1}{1-\beta}\mathbf{w}, \mathbf{c} \right\} \right], \mathbf{c} \right\}$$

where $\mathbf{w} = [0, 1, \ldots, 9]'$ and $\mathbf{c} = [c, c, \ldots, c]'$. Note that the value function will be a 10×1 matrix defined by: $V_0(\mathbf{w})^{<1>} = [V_0(w_1)^{<1>}, V_0(w_2)^{<1>}, \ldots, V_0(w_9)^{<1>}]'$. The new value function serves as a new guess (presumably closer to the solution due to the fact that the Bellman's forms a contraction) and is used to solve:

$$V_0(\mathbf{w})^{<2>} = \max_{a_t \in \{\text{accept,reject}\}} \left\{ \begin{array}{l} \mathbf{w} + \beta \mathbf{P} \left[\max_{a_{t+1} \in \{\text{accept,reject}\}} \left\{ \frac{1}{1-\beta}\mathbf{w}, \mathbf{c} + \beta \boldsymbol{\mu}' V_0(\mathbf{w})^{<1>} \right\} \right], \\ \mathbf{c} + \beta \boldsymbol{\mu}' V_0(\mathbf{w})^{<1>} \end{array} \right\}.$$

A solution is found at iteration I when the statistic

$$\frac{\left\| V_0(\mathbf{w})^{<I>} - V_0(\mathbf{w})^{<I-1>} \right\|}{(1 + \| V_0(\mathbf{w})^{<I-1>} \|)}$$

is reasonably small (where $\|\mathbf{x}\| = \sqrt{\mathbf{x}'\mathbf{x}}$). Note that a solution for this problem will be two reservation wages: a reservation wage for the first stage, $\bar{w}_0$, and for the second stage, $\bar{w}_1$.

3.5 A quantitative example

Consider the following `matlab` code below. Here we have used the dynamic programing algorithm presented above. As you can see, in the initial stage the worker always rejects wage offers below four, $\bar{w}_0 = 4$. In the second stage the worker appears to be more patient; she rejects wages offers below five, $\bar{w}_1 = 5$.

```
%
% sal1.m is an example matlab program
% to solve a simple Jovanovic model
%

% ----- save output to file ----- %

clear all
diary sal1.mout
diary off
delete sal1.mout
diary sal1.mout

% ----- display date and time of computation ----- %

%format short
date
time0 = clock;

% ----- set control vars ----- %

iters  = 200;
states = 10;

% ------ set parameters ------ %

beta = 0.95;
c    = 2*ones(states,1);

P    = [0.80 0.20 0 0 0 0 0 0 0 0;
        0.08 0.50 0.42 0 0 0 0 0 0 0;
        0.08 0.21 0.50 0.21 0 0 0 0 0 0;
        0.08 0 0.21 0.50 0.21 0 0 0 0 0;
        0.08 0 0 0.21 0.50 0.21 0 0 0 0;
        0.08 0 0 0 0.21 0.50 0.21 0 0 0;
        0.08 0 0 0 0 0.21 0.50 0.21 0 0;
        0.08 0 0 0 0 0 0.21 0.50 0.21 0;
        0.08 0 0 0 0 0 0 0.21 0.50 0.21;
        0.08 0 0 0 0 0 0 0 0.42 0.50 ] ;

% ----- find limiting distribution ----- %

[v,d]=eig(P') ;

% - find the row of the unit eigenvalue - %

[d1,d2]=find(fix(diag(d)+0.000001))

% - normalize eigenvector - %

mu = abs(v(:,d1))./sum(abs(v(:,d1)))

% ----- discretize wage ----- %

w = zeros(states,1);

for j = 1:(states-1);
 w(j+1,1) = w(j,1) + 1;
end;

% ----- value function iteration ----- %

% - initial guess of value function - %
```

```
v0  = zeros(states,1);

% - start iterations - %

crit1 = 3;

for j = 1:iters ;
 if crit1 < 0.0001, break, end;

% - compute second period payoffs and policy - %

  payoff11 = 1/(1-beta)*w ;
  payoff12 = c + beta*mu'*v0 ;
  payoff1  = [payoff11  payoff12 ];

  [v1,policy1] = max(payoff1') ;

% - put matrices in correct dimension - %

  v1      = v1';
  policy1 = policy1';

% - compute the first period payoffs and policy - %

  payoff01 = w + beta*P*v1;
  payoff02 = c + beta*mu'*v0 ;
  payoff0  = [payoff01  payoff02 ];

  [newv0,policy0] = max(payoff0') ;

% - put matrices back to correct dimension - %

  newv0   = newv0';
  policy0 = policy0';

 iter = j;

% - compute criterion - %

 crit1 = sqrt((newv0-v0)'*(newv0-v0))/(1+sqrt(v0'*v0)) ;

% - update - %

 v0 = newv0;

end;

% ----- print solutions ----- %

[w,v0,policy0,v1,policy1],  [crit1, iter]

w0bar = sum(any(policy0-1,2))
w1bar = sum(any(policy1-1,2))

comptime = etime(clock, time0)
diary off
```

```

ans =

06-Sep-2008


d1 =

     1


d2 =

     1


mu =

    0.2857
    0.2107
    0.2295
    0.1250
    0.0682
    0.0373
    0.0207
    0.0119
    0.0078
    0.0033


ans =

         0   88.1576    2.0000   88.1576    2.0000
    1.0000   88.1576    2.0000   88.1576    2.0000
    2.0000   88.1576    2.0000   88.1576    2.0000
    3.0000   88.1576    2.0000   88.1576    2.0000
    4.0000   90.1123    1.0000   88.1576    2.0000
    5.0000  100.7274    1.0000  100.0000    1.0000
    6.0000  117.5800    1.0000  120.0000    1.0000
    7.0000  136.0600    1.0000  140.0000    1.0000
    8.0000  154.5400    1.0000  160.0000    1.0000
    9.0000  165.0400    1.0000  180.0000    1.0000


ans =

    0.0001   43.0000


w0bar =

     4


w1bar =

     5


comptime =

    0.0160

```

Question 10: *Suppose that the transition probability matrix is a* 10×10 *matrix:*

$$\mathbf{P} = \begin{bmatrix} 1/10 & 1/10 & \cdots & 1/10 \\ 1/10 & 1/10 & \cdots & 1/10 \\ \vdots & \vdots & \ddots & \vdots \\ 1/10 & 1/10 & \cdots & 1/10 \end{bmatrix}.$$

Compute the reservation wages for the first stage, $\bar{w}_0$*, and for the second stage,* $\bar{w}_1$*.*

Question 11: *Suppose instead that the discount factor is* $\beta = 0.5$*. Predict what would happen to the reservation wages. Then, compute the reservation wages for the first stage,* $\bar{w}_0$*, and for the second stage,* $\bar{w}_1$*.*

4 Simulating DCDP Models

We may want to simulate the model so as to get a more complete understanding of how our agent makes her choices. For simulation, the essential elements are the reservation wages, $\bar{w}_0$ and $\bar{w}_1$, and her list of previous actions. For example, if $a_t = 2$, then w_{t+1} is drawn from the unconditional distribution $\boldsymbol{\mu}$; the `matlab` file `discreteinvrnd` draws from this distribution. Alternatively, if $a_t = 1$, then w_{t+1} is drawn from the conditional distribution $\mathbf{P}$; the `matlab` file `markov` draws from this distribution.

```
%
% sal2.m is an example matlab program
% to simulate a simple Jovanovic model
%

% ----- save output to file ----- %

clear all
diary sal2.mout
diary off
delete sal2.mout
```

```
diary sal2.mout

% ----- display date and time of computation ----- %

%format short
date
time0 = clock;

% ----- set control vars ----- %

iters  = 200;
states = 10;

% ------ set parameters ------ %

beta = 0.95;
c    = 2*ones(states,1);
P    = [0.80 0.20 0 0 0 0 0 0 0 0;
        0.08 0.50 0.42 0 0 0 0 0 0 0;
        0.08 0.21 0.50 0.21 0 0 0 0 0 0;
        0.08 0 0.21 0.50 0.21 0 0 0 0 0;
        0.08 0 0 0.21 0.50 0.21 0 0 0 0;
        0.08 0 0 0 0.21 0.50 0.21 0 0 0;
        0.08 0 0 0 0 0.21 0.50 0.21 0 0;
        0.08 0 0 0 0 0 0.21 0.50 0.21 0;
        0.08 0 0 0 0 0 0 0.21 0.50 0.21;
        0.08 0 0 0 0 0 0 0 0.42 0.50 ] ;

% ----- find limiting distribution ----- %

[v,d]=eig(P') ;

% - find the row of the unit eigenvalue - %

[d1,d2]=find(fix(diag(d)+0.000001))

% - normalize eigenvector - %

mu = abs(v(:,d1))./sum(abs(v(:,d1)))

% ----- discretize wage ----- %

w = zeros(states,1);

for j = 1:(states-1)
 w(j+1,1) = w(j,1) + 1;
end

% ----- policy solutions ----- %

w0bar = 4;
w1bar = 5;

% ----- periods and people ----- %

t = 40
n = 500

% ------ draw an initial wage for all agents ----- %

wage=discreteinvrnd(mu,n,1)-1 ;

% ----- optimal policy conditional on wage ----- %

for j = 1:n
```

```
 if wage(j,1)   >= w0bar
  policy(j,1) = 1 ;
  wageob(j,1) = wage(j,1);
 else
  policy(j,1) = 2;
  wageob(j,1) = 0;
 end
end

% ----- draw a wage conditional on previous policy ----- %

for j = 1:n
 if policy(j,1) == 1
  temp = markov(P,3,wage(j,1)+1,0:9) ;
  wage(j,1+1) = temp(2);
 else
  wage(j,1+1) = discreteinvrnd(mu,1,1)-1 ;
 end
end

% ----- optimal policy conditional on wage and previous policy ----- %

for j = 1:n
 if policy(j,1) == 1 & wage(j,1+1) >= w1bar
  policy(j,1+1) = 1 ;
  wageob(j,1+1) = wage(j,1+1);
 elseif policy(j,1) == 2 & wage(j,1+1) >= w0bar
  policy(j,1+1) = 1 ;
  wageob(j,1+1) = wage(j,1+1);
 else
  policy(j,1+1) = 2;
  wageob(j,1+1) = 0;
 end
end

% ----- now automate the above algorithm ----- %

for i = 1:(t-2)

% - draw a wage conditional on previous policy - %

for j = 1:n
 if policy(j,1+i) == 1 & policy(j,i) == 1
  wage(j,1+1+i) = wage(j,1+i);
 elseif policy(j,1+i) == 1 & policy(j,i) == 2
  temp=markov(P,3,wage(j,1+i)+1,0:9) ;
  wage(j,1+1+i) = temp(2);
 else
  wage(j,1+1+i) = discreteinvrnd(mu,1,1)-1 ;
 end
end

% - optimal policy conditional on wage and previous policy - %

for j = 1:n
 if policy(j,1+i) == 1 & wage(j,1+1+i) >= w1bar
  policy(j,1+1+i) = 1 ;
  wageob(j,1+1+i) = wage(j,1+1+i);
 elseif policy(j,1+i) == 2 & wage(j,1+1+i) >= w0bar
   policy(j,1+1+i) = 1 ;
   wageob(j,1+1+i) = wage(j,1+1+i);
 else
  policy(j,1+1+i) = 2;
  wageob(j,1+1+i) = 0;
 end
```

```
end

end

% ----- compute labor force participation rate ----- %

lfp = mean((1./policy-1/2)*2) ;

% ----- print a typical simulated agents wage and observed wage ----- %

[wage(14,1:t)' wageob(14,1:t)' policy(14,1:t)']

comptime = etime(clock, time0)
diary off
```

```

ans =

03-Sep-2008


d1 =

     1


d2 =

     1


mu =

    0.2857
    0.2107
    0.2295
    0.1250
    0.0682
    0.0373
    0.0207
    0.0119
    0.0078
    0.0033


t =

    40


n =

   500


ans =

```

```
        1     0     2
        1     0     2
        0     0     2
        3     0     2
        4     4     1
        0     0     2
        0     0     2
        3     0     2
        1     0     2
        0     0     2
        3     0     2
        3     0     2
        1     0     2
        0     0     2
        0     0     2
        0     0     2
        0     0     2
        8     8     1
        7     7     1
        7     7     1
        7     7     1
        7     7     1
        7     7     1
        7     7     1
        7     7     1
        7     7     1
        7     7     1
        7     7     1
        7     7     1
        7     7     1
        7     7     1
        7     7     1
        7     7     1
        7     7     1
        7     7     1
        7     7     1
        7     7     1
        7     7     1
        7     7     1
        7     7     1


comptime =

    0.7500

```

Question 12: *Compute the average observed wage for each time period. Is it increasing or decreasing overtime?*

Question 13: *Re-simulate the model for where $\beta = 0.5$. Then, compute the average observed wage for each time period. Is the average wage lower or higher in this economy?*

Question 14: *Compute the sample's average labor force participation rate for each time* *period. Is it increasing or decreasing overtime?*

Question 15: *Re-simulate the model for where $c = 4$. Then, compute the sample's average labor force participation rate for each time period. Is labor force participation lower or higher in this economy?*

5 Choice probabilities

Suppose that our worker's first stage reservation wage is four (she doesn't work if her offer is below four, $\bar{w}_0 = 4$) and we observe her working at her job for an initial six dollars, $w_t^{obs} = 6$, at time t. We may want to know the probability of her landing a six dollars an hour job given that she accepted it.

More specifically, we want to compute

$$\Pr\left\{w_t^{obs} = 6 | w_t \geq 4\right\} = \Pr\left\{w_t^{obs} = 6 | w_t \geq \bar{w}_0\right\} = \Pr\left\{w_t^{obs} = 6 | a_t = 1\right\}.$$

This statistic is easily computed by: $\frac{\mu_7}{\mu_5 + \mu_6 + \cdots + \mu_{10}}$. In the previous numerical example this number is:

$$\begin{aligned} \Pr\left\{w_t^{obs} = 6 | a_t = 1\right\} &= \frac{0.0207}{0.0682 + 0.0373 + 0.0207 + 0.0119 + 0.0078 + 0.0033} \\ &= 0.1387. \end{aligned}$$

More formally, for observed workers in the first stage, the probability that they received a particular wage is defined as:

$$\delta_0(i) = \Pr\{w_t^{obs} = w_i | a_t = 1\} = \frac{\mu_{i+1}}{\sum\limits_{j > \bar{w}_0} \mu_j}.$$

These probabilities are computed by `matlab` as:

$$\delta_0 = \{0, 0, 0, 0, 0.4571, 0.2502, 0.1387, 0.0801, 0.0520, 0.0218\}.$$

The probability that a time period $t+1$ wage w_{t+1} will be offered given that the agent accepted the initial wage offer is defined as $\Pr\left\{ w_{t+1}^{obs} = w_i \middle| a_t = 1\right\}$. It can be computed as:

$$\gamma_1(i) = \Pr\left\{ w_{t+1}^{obs} = w_i \middle| a_t = 1\right\} = \sum_{j>\bar{w}_0} \delta_0(j) P_{j,i}.$$

Next, the probability that an agent will be working at time period $t+1$ for a wage w_{t+1} given that the agent accepted the initial wage offer and the second offer is given by:

$$\begin{aligned}
\delta_1(i) &= \Pr\left\{ w_{t+1}^{obs} = w_i \middle| w_t \geq \bar{w}_0 \text{ and } w_{t+1} \geq \bar{w}_1\right\} \\
&= \Pr\left\{ w_{t+1}^{obs} = w_i \middle| a_t = 1 \text{ and } a_{t+1} = 1\right\} \\
&= \frac{\gamma_1(i)}{\sum\limits_{j>\bar{w}_1} \gamma_1(j)}.
\end{aligned}$$

These probabilities are computed by `matlab` as:

$$\delta_1 = \{0, 0, 0, 0, 0, 0.4556, 0.2526, 0.1458, 0.0863, 0.0596\}.$$

The probability that an agent will be working at time period $t+2$ for a wage w_{t+2} given that the agent accepted the initial wage offer and the second offer is given by:

$$\begin{aligned}
\gamma_2(i) &= \Pr\left\{ w_{t+2}^{obs} = w_i \middle| w_t \geq \bar{w}_0 \text{ and } w_{t+1} \geq \bar{w}_1\right\} \\
&= \Pr\left\{ w_{t+2}^{obs} = w_i \middle| a_t = 1 \text{ and } a_{t+1} = 1\right\} \\
&= \sum_{j>\bar{w}_1} \delta_1(j) I_{j,i}.
\end{aligned}$$

where I is a 10×10 identity matrix (basically after the second period the wage doesn't transition anymore). Finally, the probability that an agent will be working at time period $t+2$ for a wage w_{t+2} given that the agent accepted the initial wage offer, the second offer, and the third offer, $\Pr\left\{ w_{t+2}^{obs} = w_i \middle| w_t \geq \bar{w}_0 \text{ and } w_{t+1} \geq \bar{w}_1 \text{ and } w_{t+2} \geq \bar{w}_1 \right\}$, is given by:

$$
\begin{aligned}
\delta_2(i) &= \Pr\left\{ w_{t+2}^{obs} = w_i \middle| a_t = 1 \text{ and } a_{t+1} = 1 \text{ and } a_{t+2} = 1 \right\} \\
&= \frac{\gamma_2(i)}{\sum\limits_{j > \bar{w}_1} \gamma_2(j)}.
\end{aligned}
$$

These probabilities are computed by `matlab` as:

$$
\delta_2 = \{0, 0, 0, 0, 0, 0.4556, 0.2526, 0.1458, 0.0863, 0.0596\}.
$$

Notice that $\delta_1 = \delta_2 = \ldots = \delta_\infty$.

```
%
% sal3.m is an example matlab program
% to compute choice probabilities
%

% ----- save output to file ----- %

clear all
diary sal3.mout
diary off
delete sal3.mout
diary sal3.mout

% ----- display date and time of computation ----- %

%format short
date
time0 = clock;

% ----- set control vars ----- %

iters  = 200;
states = 10;

% ------ set parameters ------ %

beta = 0.95;
c    = 2*ones(states,1);
P    = [0.80 0.20 0 0 0 0 0 0 0 0;
        0.08 0.50 0.42 0 0 0 0 0 0 0;
        0.08 0.21 0.50 0.21 0 0 0 0 0 0;
```

```
         0.08 0 0.21 0.50 0.21 0 0 0 0 0;
         0.08 0 0 0.21 0.50 0.21 0 0 0 0;
         0.08 0 0 0 0.21 0.50 0.21 0 0 0;
         0.08 0 0 0 0 0.21 0.50 0.21 0 0;
         0.08 0 0 0 0 0 0.21 0.50 0.21 0;
         0.08 0 0 0 0 0 0 0.21 0.50 0.21;
         0.08 0 0 0 0 0 0 0 0.42 0.50 ] ;

% ----- find limiting distribution ----- %

[v,d]=eig(P') ;

% - find the row of the unit eigenvalue - %

[d1,d2]=find(fix(diag(d)+0.000001))

% - normalize eigenvector - %

mu = abs(v(:,d1))./sum(abs(v(:,d1)))

% ----- solutions ----- %

w0bar = 4
w1bar = 5

% ----- compute chice probabilities ------ %

for j = (w0bar+1):states ;
 delta0(j,1)=mu(j,1)/sum(mu(5:10,1)) ;
end

gamma1 = P*delta0 ;

for j = (w1bar+1):states ;
 delta1(j,1)=gamma1(j,1)/sum(gamma1(6:10,1)) ;
end

gamma2 = eye(10)*delta1;

for j = (w1bar+1):states ;
 delta2(j,1)=gamma2(j,1)/sum(gamma2(6:10,1)) ;
end

[delta0 delta1 delta2]

comptime = etime(clock, time0)
diary off
```

```

ans =

03-Sep-2008


d1 =

     1
```

```


d2 =

     1


mu =

    0.2857
    0.2107
    0.2295
    0.1250
    0.0682
    0.0373
    0.0207
    0.0119
    0.0078
    0.0033


w0bar =

     4


w1bar =

     5


ans =

         0         0         0
         0         0         0
         0         0         0
         0         0         0
    0.4571         0         0
    0.2502    0.4556    0.4556
    0.1387    0.2526    0.2526
    0.0801    0.1458    0.1458
    0.0520    0.0863    0.0863
    0.0218    0.0596    0.0596


comptime =

    0.0160

```

6 Computing Likelihoods: An Estimation Algorithm

Suppose that we observed the following set of wages $\{0, 0, 0, 0, 5, 6, 6, 6\}$ generated from the matching model. We want to know, given our model, what is the probability we observed such an outcome? More specifically, we define the likelihood of this event as:

$$\mathcal{L} = \Pr\left\{w_t^{obs} = 0, w_{t+1}^{obs} = 0, w_{t+2}^{obs} = 0, w_{t+3}^{obs} = 0, w_{t+4}^{obs} = 5, w_{t+5}^{obs} = 6, w_{t+6}^{obs} = 6, w_{t+7}^{obs} = 6\right\}.$$

Because the first five wages were drawn from the unconditional distribution, they are uncorrelated. This allows us to write the likelihood as:

$$\begin{aligned}\mathcal{L} &= \Pr\left\{w_t^{obs} = 0\right\}\Pr\left\{w_{t+1}^{obs} = 0\right\}\Pr\left\{w_{t+2}^{obs} = 0\right\}\times \\ &\quad \Pr\left\{w_{t+3}^{obs} = 0\right\}\Pr\left\{w_{t+4}^{obs} = 5\right\}\Pr\left\{w_{t+5}^{obs} = 6, w_{t+6}^{obs} = 6, w_{t+7}^{obs} = 6\right\}\end{aligned}$$

The sixth wage, w_{t+5}^{obs}, was drawn from the conditional distribution: condition on $w_{t+4}^{obs} = 5$. The seventh wage, w_{t+6}^{obs}, was drawn from the conditional distribution: conditional on $w_{t+5}^{obs} = 6$ and $w_{t+4}^{obs} = 5$. Finally, the seventh wage, w_{t+7}^{obs}, was drawn from the conditional distribution: conditional on $w_{t+6}^{obs} = 6$ and $w_{t+5}^{obs} = 6$. This allows us to write the likelihood as:

$$\begin{aligned}\mathcal{L} &= \Pr\left\{w_t^{obs} = 0\right\}\Pr\left\{w_{t+1}^{obs} = 0\right\}\Pr\left\{w_{t+2}^{obs} = 0\right\}\times \\ &\quad \Pr\left\{w_{t+3}^{obs} = 0\right\}\Pr\left\{w_{t+4}^{obs} = 5\right\}\Pr\left\{w_{t+5}^{obs} = 6 | w_{t+4}^{obs} = 5\right\}\times \\ &\quad \Pr\left\{w_{t+6}^{obs} = 6 | w_{t+5}^{obs} = 6, w_{t+4}^{obs} = 5\right\}\Pr\left\{w_{t+7}^{obs} = 6 | w_{t+6}^{obs} = 6, w_{t+5}^{obs} = 6\right\}\end{aligned}$$

It is easy to show that $\Pr\left\{w_{t+j}^{obs} = 0\right\} = \mu_1 + \mu_2 + \mu_3 + \mu_4 = 0.7258$ for all j. And, $\Pr\left\{w_{t+4}^{obs} = 5\right\} = \mu_6 = 0.0373$, $\Pr\left\{w_{t+5}^{obs} = 6 | w_{t+4}^{obs} = 5\right\} = P_{6,7} = 0.21$. Finally, $\Pr\{w_{t+6}^{obs} = 6 | w_{t+5}^{obs} = 6, w_{t+4}^{obs} = 5\} = 1$ and $\Pr\{w_{t+7}^{obs} = 6 | w_{t+6}^{obs} = 6, w_{t+5}^{obs} = 6\} = 1$. Therefore, the likelihood is:

$$\mathcal{L} = 0.7258 \cdot 0.7258 \cdot 0.7258 \cdot 0.7258 \cdot 0.0373 \cdot 0.21 \cdot 1 \cdot 1 = 2.1737 \times 10^{-3}$$

Question 16: *Compute the likelihood of the events* $\{0,0,4,4,4,4,4,4\}$, $\{0,0,5,0,6,7,7,7\}$, *and* $\{0,0,0,0,6,7,7,7\}$.

Question 17: *Suppose instead you believed (wrongly) that the transition probability matrix was given by the* 10×10 *transition probability matrix:*

$$\mathbf{P} = \begin{bmatrix} 1/10 & 1/10 & \cdots & 1/10 \\ 1/10 & 1/10 & \cdots & 1/10 \\ \vdots & \vdots & \ddots & \vdots \\ 1/10 & 1/10 & \cdots & 1/10 \end{bmatrix}.$$

Compute the likelihood of the event $\{0,0,0,0,6,7,7,7\}$.

Question 18: *Suppose instead you believed (wrongly) that the transition probability matrix was given by the* 10×10 *transition probability matrix:*

$$\mathbf{P} = \begin{bmatrix} 1/10 & 1/10 & \cdots & 1/10 \\ 1/10 & 1/10 & \cdots & 1/10 \\ \vdots & \vdots & \ddots & \vdots \\ 1/10 & 1/10 & \cdots & 1/10 \end{bmatrix}.$$

Compute the likelihood of the event $\{0,0,0,0,5,6,6,6\}$.

A second use for computing likelihoods is for estimating the parameters of a model. Notice that the likelihood of observing the event $\{0,0,0,0,5,6,6,6\}$ goes down for alternative (and wrong) values for $\mathbf{P}$. Maximum likelihood chooses the model parameters, like $\theta = \{\mathbf{P}, c, \beta\}$, to make $\mathcal{L}$ as large as possible. We define a maximum likelihood estimator by:

$$\hat{\theta} = \max_{\theta}\{\mathcal{L}(\theta)\}.$$

A solution to the maximum likelihood problem is where:

$$\frac{\partial \mathcal{L}(\hat{\theta})}{\partial \theta} = 0.$$

Much more later ...

7 Counterfactual Experiments

Panel (A) of Figure 1 shows a time series of labor force participation rates for two sets of cohorts. The first cohort are those born in 1945 and begin to work by 1965. The second set of cohorts are born in 1965 and start work in the 80's. Notice the later cohort appear to be lazier; they participate in the labor market at much lower rates than their parents.

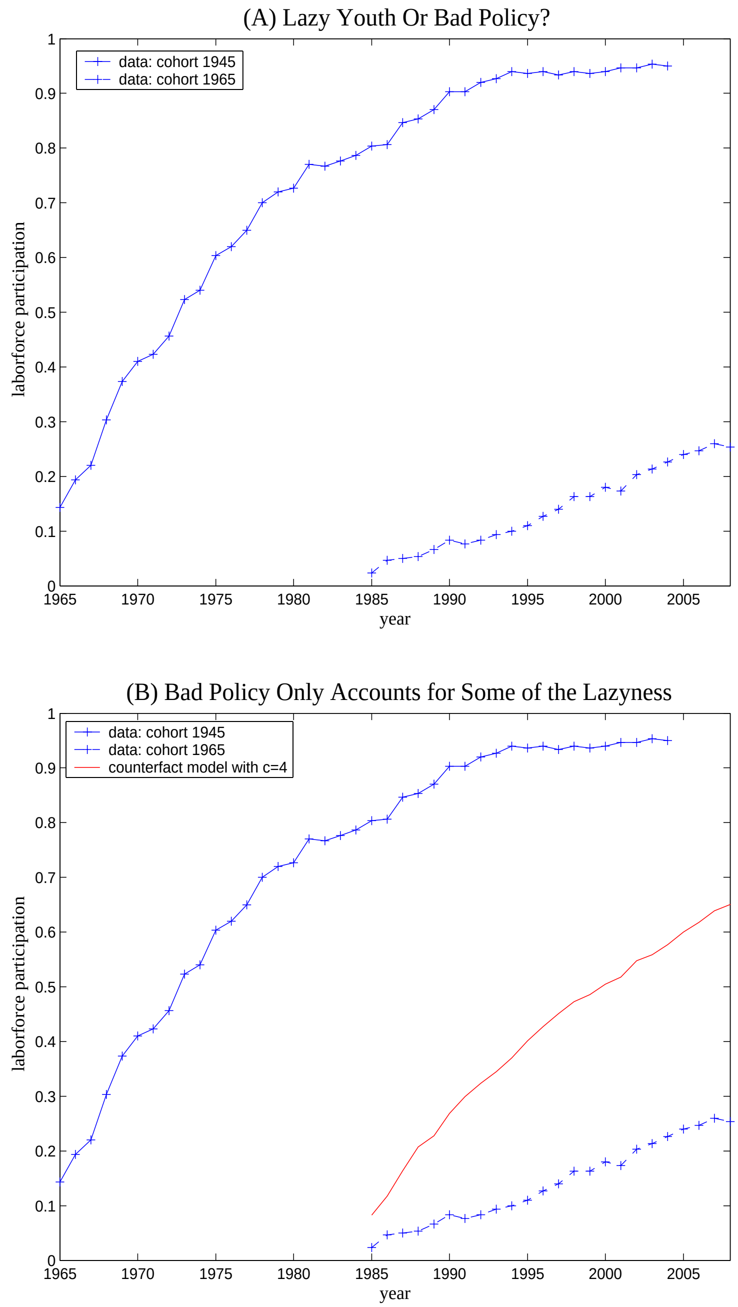
As researchers, we may have estimated the parameters of the model – via maximum likelihood – on the 1945 cohort data. This allows us to test different theories about the fall in youth labor force participation. One such theory is credited to the government making unemployment insurance more generous in the 1980's (assumed too late to affect the 1945 cohort). More specifically, suppose that we know the government increased unemployment compensation from $c = 2$ to $c = 4$. The Panel (B) of Figure 1 shows that this theory can account for some (roughly 50%) of the fall in youth participation. We would say it is a tentatively good but incomplete theory.

Question 19: *Suppose research has shown that the government reformed welfare in the 1980's. All forms of unemployment compensation were completely eliminated;* $c = 0$*. In its place, suppose the government imposed a minimum wage of* $w^{\min} = 4$*. The firm may offer only wages zero (basically no offer) or greater than or equal to four. Essentially, the state space becomes* $w_i \in \{0, 4, 5, \ldots, 9\}$*. Further, assume the match* *evolves according to the transition probability matrix:*

$$\begin{bmatrix} P_{1,1} & P_{1,2} & 0 & 0 & 0 & 0 & 0 \\ P_{2,1} & P_{2,2} & P_{2,3} & 0 & 0 & 0 & 0 \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} & 0 & 0 & 0 \\ P_{4,1} & 0 & P_{4,3} & P_{4,4} & P_{4,5} & 0 & 0 \\ P_{5,1} & 0 & 0 & P_{5,4} & P_{5,5} & P_{5,6} & 0 \\ P_{6,1} & 0 & 0 & 0 & P_{6,5} & P_{6,6} & P_{6,7} \\ P_{7,1} & 0 & 0 & 0 & 0 & P_{7,6} & P_{7,7} \end{bmatrix} = \begin{bmatrix} .8 & .2 & 0 & 0 & 0 & 0 & 0 \\ .08 & .5 & .42 & 0 & 0 & 0 & 0 \\ .08 & .21 & .5 & .21 & 0 & 0 & 0 \\ .08 & 0 & .21 & .5 & .21 & 0 & 0 \\ .08 & 0 & 0 & .21 & .5 & .21 & 0 \\ .08 & 0 & 0 & 0 & .21 & .5 & .21 \\ .08 & 0 & 0 & 0 & 0 & .42 & .5 \end{bmatrix}$$

where $P_{2,1} = \Pr\{w_{t+1} = 0 | w_t = 4\}$, *for example. Note that wages equal to one, two, and three have been completely eliminated from* $\mathbf{P}$*. Compute the reservation wages (assuming matching). Then, simulate a sample of labor force participation rates. Can this policy change account for the declining youth rate?*

Figure 1: Counterfactual Experiments



8 Generalized DCDP Models: An Application to Real Estate

The previous examples are intended to show simple situations where agents make dynamic discrete choices. In these simple cases, wages are exogenously offered to the worker. It is this unfortunate feature that makes these models not that helpful in economic analysis. To be specific, as economists we know that wages are not exogenously offered but are determined, in part, by the demographic features of the worker. The demographic features of interest in economics include race, education, experience, and so on. As economists, we also know that their are potential differences in how individuals form utility from their actions. Given the same choices, consumers often choose different actions just because of difference in preferences. Another way to say this is that we all have different reservation wages.

Generalized DCDP models allow for a richer interaction along these lines and are nicely reviewed in Keane and Wolpin (2008), "Empirical applications of discrete choice dynamic programming models." The idea of most of these models is to add the equilibrium idea of supply and demand and heterogeneity in preferences. On the demand side – such as the demand for labor, a demand function is typically posited into the model. The worker then has to decide to supply labor or engage in other activities (such as schooling) given unobserved (unobserved by us) preferences for each action. The equilibrium is where supply and demand equate.

The first example of a generalized DCDP model is as an application to real estate. Following Yinger (1981)[2], I model two types of uncertainty facing sellers of real estate: (i) uncertainty about the number of buyers of housing and thus offers; and (ii) uncertainty about matches between buyers and listings.

[2] Yinger, John (1981), "A Search Model of Real Estate Broker Behavior," *The American Economic Review* **71(4)**, pp. 591-605.

8.1 Housing demand

Each household assumes that the true value of their house is determined by its attributes. More specifically, the i'th seller at time t faces a hedonic pricing formula (demand) for their property given by:

$$\begin{aligned} \ln P_{i,t} &= \beta_0 + \beta_1 SQ_i + \beta_2 Bath_i + \beta_3 Neigh_i + \ldots + \varepsilon_{i,t} \\ &= \mathbf{x}_i \boldsymbol{\beta} + \varepsilon_{i,t} \end{aligned}$$

where $\varepsilon_{i,t} \sim N(0, \sigma_\varepsilon^2)$ and $\mathbf{x}_i$ is a vector of housing characteristics. Here, $\varepsilon_{i,t}$ determines – in part – the uncertainty about the offer of the potential buyer. The $\mathbf{x}_i \boldsymbol{\beta}$ represents the deterministic value of the house.

8.2 The Bellman's equation

If the seller accepts a time t offer of $P_{i,t}$, she then receives a random utility component $\bar{\epsilon}^{<1>} + \epsilon_{i,t}^{<1>}$. Upon acceptance of a contract, the seller pays a fraction of the total price, c, as commission to the real estate broker. The value of accepting an offer (denoted $a_{i,t} = 1$) at the optimal choice $a_{i,t}$, net of the listing price, is:

$$V^{<1>}(\Omega_{i,t}) = (1-c)P_{i,t} - \bar{P}_i + \bar{\epsilon}_i^{<1>} + \epsilon_{i,t}^{<1>}$$

where $\Omega_{i,t}$ is state space faced by the household. It is important to distinguish the role $\bar{\epsilon}_i^{<1>} + \epsilon_{i,t}^{<1>}$ plays in determination of the utility from a choice to accept the offer. In some real estate deals, closing costs may be paid by the seller; in this case closing costs could be included $\bar{\epsilon}_i^{<1>}$. Typically, these costs are not identified by the econometrician and are, thus, modelled as and unobserved random component. As well, the household may receive utility just from the fact that their house was sold and thus not measurable by the econometrician.

In any event, $\bar{\epsilon}_i^{<1>} + \epsilon_{i,t}^{<1>}$ is to capture all direct but unobservable utility received from the sale of the property.

Alternatively, if the home owner rejects an offer, she pays – both indirectly and directly – a showing cost. Indirect showing costs results from the lost utility from having not sold the house and, for example, having strangers within the house. A direct showing cost includes the necessary expenditures needed to keep the house in showing condition. Total utility lost from rejecting the offer is denoted $\bar{\epsilon}_i^{<2>} + \epsilon_{i,t}^{<2>}$. Additionally, if the seller rejects an offer greater than the listing price, she is obligated to pay the lost commission to the real estate broker:

$$LC_{i,t} = I(P_{i,t} - \bar{P}_i \geq 0) \cdot c \cdot P_{i,t}$$

where $I(\cdot)$ is an indicator function that is 1 where it's argument is true and zero elsewhere. The value of rejecting an offer (denoted $a_{i,t} = 2$) is:

$$V^{<2>}(\Omega_{i,t}) = -LC_{i,t} + \bar{\epsilon}_i^{<2>} + \epsilon_{i,t}^{<2>} + \beta E_t V(\Omega_{i,t+1}).$$

Finally, the household solves the Bellman's equation:

$$\begin{aligned}
V(\Omega_{i,t}) &= \max_{a_t \in \{\text{accept},\text{reject}\}} \left\{ V^{<1>}(\Omega_{i,t}), V^{<2>}(\Omega_{i,t}) \right\} \\
&= \max_{a_t \in \{\text{accept},\text{reject}\}} \left\{ \begin{array}{l} (1-c)P_{i,t} - \bar{P}_i + \bar{\epsilon}_i^{<1>} + \epsilon_{i,t}^{<1>}, \\ -LC_{i,t} + \bar{\epsilon}_i^{<2>} + \epsilon_{i,t}^{<2>} + \beta E_t V(\Omega_{i,t+1}) \end{array} \right\} \\
&= \max_{a_t \in \{\text{accept},\text{reject}\}} \left\{ \begin{array}{l} (1-c)\exp(\mathbf{x}_i\boldsymbol{\beta} + \varepsilon_{i,t}) - \bar{P}_i + \bar{\epsilon}_i^{<1>} + \epsilon_{i,t}^{<1>}, \\ \left[\begin{array}{c} -I(\exp(\mathbf{x}_i\boldsymbol{\beta} + \varepsilon_{i,t}) - \bar{P}_i \geq 0) \cdot c \cdot \exp(\mathbf{x}_i\boldsymbol{\beta} + \varepsilon_{i,t}) + \\ \bar{\epsilon}_i^{<2>} + \epsilon_{i,t}^{<2>} + \beta E_t V(\Omega_{i,t+1}) \end{array} \right] \end{array} \right\}
\end{aligned}$$

8.3 Setting up the state space

Solving the model can be accomplished by value function iteration. Before the equilibrium can be solved, however, the state space must be defined. Because the state space is large, however, the setup and resulting search over the space is rather complicated. To see this note that the set $\Omega_{i,t}$ includes the list[3] of variables $\{\mathbf{x}_i, \varepsilon_{i,t}, , \epsilon_{i,t}^{<1>}, \epsilon_{i,t}^{<2>}\}$, of which a solution needs to be found for every combination (permuation). These multiple combinations lead to a "curse of dimensionality."

For example, suppose that value is determined from two components: square-footage and number of baths. The square footage can naturally be discreteized into discrete units: let $SQ_i = \{1, 2, 3\}$ for small (0-1500 sqft), medium (1500-3500 sqft), and large homes (3500 sqft and larger). Additionally, the number of baths are easly discreteized as well: $Bath_i = \{1, 2, 3\}$. If the state space included only these states then we would have to solve over every permuation of $\{1, 2, 3\}$. Specifically,

$$\Omega_{i,t} = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \end{bmatrix}'.$$

where the first column is the square footage and the second is the number of baths – there are nine possible states.

Now suppose that we included $\varepsilon_{i,t}$, $\epsilon_{i,t}^{<1>}$, and $\epsilon_{i,t}^{<2>}$ in the state space. A natural way to discretized these continuous random variables is to use a quadrature method. The matlab file `qnwnorm.m` discretize a normally distributed random variable with mean μ and variance σ^2 into n distinct nodes. For example, let use discretize $\epsilon_{i,t}^{<1>}$ by assuming $n = 3$, $\mu = 0$, and $\sigma^2 = 0.4$; the program statement `[s,p]=qnwnorm(3,0,.4)` returns and `s` matrix of n distinct nodes and `p` matrix of probabilities of the node occurring.

[3] We have assumed that $\bar{\epsilon}_i^{<1>} = \bar{\epsilon}_i^{<2>} = 0$.

```
>> [s,p]=qnwnorm(3,0,.4)
ans =
   -1.0954    0.1667
         0    0.6667
    1.0954    0.1667
```

Given that we have discretized all of our unobserved random variables via the quadrature method, we now want to combine our state variables so that every permutation is considered. Let $SQ_i = [1,2,3]'$, $Bath_i = [1,2,3]'$, $\varepsilon_{i,t} = [-1.0954, 0, 1.0954]'$, $\epsilon_{i,t}^{<1>} = [-1.0954, 0, 1.0954]'$, and $\epsilon_{i,t}^{<2>} = [-1.0954, 0, 1.0954]'$. The matlab file `gridmake.m` does just that. The command `omega=gridmake(sq,bath,epsi,e1,e2)` returns a 243×5 matrix with every permutation possible; this is the state matrix Ω_i that defines our value functions. The first few lines of the output are:

```
>> omega=gridmake(sq,bath,epsi,e1,e2);
>> omega(1:20,:) %display the first twenty rows
ans =
    1.0000    1.0000   -1.0954   -1.0954   -1.0954
    2.0000    1.0000   -1.0954   -1.0954   -1.0954
    3.0000    1.0000   -1.0954   -1.0954   -1.0954
    1.0000    2.0000   -1.0954   -1.0954   -1.0954
    2.0000    2.0000   -1.0954   -1.0954   -1.0954
    3.0000    2.0000   -1.0954   -1.0954   -1.0954
    1.0000    3.0000   -1.0954   -1.0954   -1.0954
    2.0000    3.0000   -1.0954   -1.0954   -1.0954
    3.0000    3.0000   -1.0954   -1.0954   -1.0954
    1.0000    1.0000         0   -1.0954   -1.0954
    2.0000    1.0000         0   -1.0954   -1.0954
    3.0000    1.0000         0   -1.0954   -1.0954
    1.0000    2.0000         0   -1.0954   -1.0954
    2.0000    2.0000         0   -1.0954   -1.0954
    3.0000    2.0000         0   -1.0954   -1.0954
    1.0000    3.0000         0   -1.0954   -1.0954
    2.0000    3.0000         0   -1.0954   -1.0954
    3.0000    3.0000         0   -1.0954   -1.0954
    1.0000    1.0000    1.0954   -1.0954   -1.0954
```

```
    2.0000    1.0000    1.0954   -1.0954   -1.0954
```

With the state space defined, the next step is to construct a probability matrix so that the expectation $E_t V(\Omega_{i,t+1})$ can be computed. Suppose that we are a home owner with one bath and 1,000 sqft ($SQ = 1$ and $Bath = 1$). The current choice (sell or hold) depends on our expectations of future events (such as future better offers). However, some future events will never occur; our house will never mutate to a two bath 9,500 sqft home in the future (still the same old $SQ = 1$ and $Bath = 1$ house). Therefore, the probability of those states occurring must be zero. The following code computes a transition matrix consistent with these ideas:

```
% ----- define states ----- %

sq   = [1;2;3] ;
bath = [1;2;3];

% - construct grid for unobserved states by quadrature - %

[ep,eppr] = qnwnorm(3,0,0.4);
[e1,epr1] = qnwnorm(3,0,0.4);
[e2,epr2] = qnwnorm(3,0,0.4);

% ----- state space: find every permuation for the states ----- %

omega=gridmake(sq,bath,ep,e1,e2);

% ----- define transition probability matrix ----- %

temp = eye(3,3) ;

for j = 1:3
    for i = 1:3
     P(i,:,j) = prod(gridmake(temp(:,i),temp(:,j),eppr,epr1,epr2)') ;
 end
end
```

```
PP = [P(:,:,1) ; P(:,:,2); P(:,:,3)] ;

clear P;
P  = PP;

for j = 1:(3*3*3-1)
    P = [P;PP];
end;
```

8.4 Experimental design: solving and simulating the model

Given our defined state space and **P** matrix, value function iteration can begin once the parameters of the model are defined. For the demand, I assume the following specification:

$$\ln P_{i,t} = 6 + 1 \cdot SQ_i + 1 \cdot Bath_i + \varepsilon_{i,t}$$

Additionally, the states are discretized as before: $SQ_i = [1,2,3]'$, $Bath_i = [1,2,3]'$, $\varepsilon_{i,t} = [-1.0954, 0, 1.0954]'$, $\epsilon_{i,t}^{<1>} = [-1.0954, 0, 1.0954]'$, and $\epsilon_{i,t}^{<2>} = [-1.0954, 0, 1.0954]'$. The remaining parameters – such as c, β, and $\bar{P}_i$ – can be found in the matlab file `real1.m`.

```
%
% real1.m is an example matlab program
% to solve a simple real estate search model
%

format bank

% ----- save output to file ----- %

clear all
diary real1.mout
diary off
delete real1.mout
diary real1.mout

% ----- display date and time of computation ----- %

%format short
date
time0 = clock;

% ----- set control vars ----- %

```

```
iters  = 5700;

% ----- define parameters of the model ----- %

c    = 0.07 ;
pbar = 1.90 ;
beta = (1/1.03)^(1/12) ;

beta0 = 6;
beta1 = 1;
beta2 = 1;

% ----- define states ----- %

sq   = [1;2;3] ;
bath = [1;2;3];

% - construct grid for unobserved states by quadrature - %

[ep,eppr] = qnwnorm(3,0,0.4);
[e1,epr1] = qnwnorm(3,0,0.4);
[e2,epr2] = qnwnorm(3,0,0.4);

% ----- state space: find every permuation for the states ----- %

omega=gridmake(sq,bath,ep,e1,e2);

% ----- define transition probability matrix ----- %

temp1 = eye(3,3) ;
temp2 = eye(3,3) ;

for j = 1:3
    for i = 1:3
     P(i,:,j) = prod(gridmake(temp1(:,i),temp2(:,j),eppr,epr1,epr2)') ;
 end
end

PP = [P(:,:,1) ; P(:,:,2); P(:,:,3)] ;

clear P;
P  = PP;

for j = 1:(3*3*3-1)
    P = [P;PP];
end;

% ----- find size of state space ----- %

[states,states2] = size(omega);

% ----- compute price and listing price ----- %

price    = exp(beta0 + beta1*omega(:,1) + beta2*omega(:,2) + omega(:,3)) ;
pricebar = pbar*exp(beta0 + beta1*omega(:,1) + beta2*omega(:,2) + .4) ;

% ----- compute lost commission ----- %

for j = 1:states
  if price(j,1) >= pricebar(j,1)
   LC(j,1) = c*price(j,1);
  else
   LC(j,1) = 0;
 end
end
```

```

% ----- value function iteration ----- %

% - initial guess of value function - %

v0  = zeros(states,1);

% - start iterations - %

crit1 = 3;

for j = 1:iters ;
 if crit1 < 0.0001, break, end;

% - compute second period payoffs and policy - %

  payoff1 = (1-c)*price - pricebar + omega(:,4);
  payoff2 = -LC + omega(:,5) + beta*P*v0 ;
  payoff0 = [payoff1  payoff2 ];

  [newv0,policy0] = max(payoff0') ;

% - put matrices back to correct dimension - %

  newv0   = newv0';
  policy0 = policy0';

 iter = j;

% - compute criterion - %

 crit1 = sqrt((newv0-v0)'*(newv0-v0))/(1+sqrt(v0'*v0)) ;

% - update - %

 v0 = newv0;

end;

% ----- print solutions ----- %

[[1:states]' omega(:,1:2), price, v0, policy0],  [crit1, iter]

% ----- save data to file ----- %

X=[[1:states]' omega(:,1:5), price, v0, policy0] ;

save real1.dat X -ascii

comptime = etime(clock, time0)
diary off
```

The output shows, for example, that a homeowner with $SQ = 1$ and $Bath = 1$ would not sell their property for less than $8,914.60.

```

ans =

```

```
11-Sep-2008


ans =

          1.00          1.00          1.00        996.80       -157.55          2.00
          2.00          2.00          1.00       2709.60       -426.39          2.00
          3.00          3.00          1.00       7365.45      -1157.16          2.00
          4.00          1.00          2.00       2709.60       -426.39          2.00
          5.00          2.00          2.00       7365.45      -1157.16          2.00
          6.00          3.00          2.00      20021.36      -3143.61          2.00
          7.00          1.00          3.00       7365.45      -1157.16          2.00
          8.00          2.00          3.00      20021.36      -3143.61          2.00
          9.00          3.00          3.00      54423.69      -8543.34          2.00
         10.00          1.00          1.00       2980.96       -157.55          2.00
         11.00          2.00          1.00       8103.08       -426.39          2.00
         12.00          3.00          1.00      22026.47      -1157.16          2.00
         13.00          1.00          2.00       8103.08       -426.39          2.00
         14.00          2.00          2.00      22026.47      -1157.16          2.00
         15.00          3.00          2.00      59874.14      -3143.61          2.00
         16.00          1.00          3.00      22026.47      -1157.16          2.00
         17.00          2.00          3.00      59874.14      -3143.61          2.00
         18.00          3.00          3.00     162754.79      -8543.34          2.00
         19.00          1.00          1.00       8914.60       -159.95          1.00
         20.00          2.00          1.00      24232.38       -432.90          1.00
         21.00          3.00          1.00      65870.44      -1174.87          1.00
         22.00          1.00          2.00      24232.38       -432.90          1.00
         23.00          2.00          2.00      65870.44      -1174.87          1.00
         24.00          3.00          2.00     179054.43      -3191.75          1.00
         25.00          1.00          3.00      65870.44      -1174.87          1.00
         26.00          2.00          3.00     179054.43      -3191.75          1.00
         27.00          3.00          3.00     486720.40      -8674.19          1.00
         28.00          1.00          1.00        996.80       -157.55          2.00
         29.00          2.00          1.00       2709.60       -426.39          2.00
         30.00          3.00          1.00       7365.45      -1157.16          2.00
         31.00          1.00          2.00       2709.60       -426.39          2.00
         32.00          2.00          2.00       7365.45      -1157.16          2.00
         33.00          3.00          2.00      20021.36      -3143.61          2.00
         34.00          1.00          3.00       7365.45      -1157.16          2.00
         35.00          2.00          3.00      20021.36      -3143.61          2.00
         36.00          3.00          3.00      54423.69      -8543.34          2.00
         37.00          1.00          1.00       2980.96       -157.55          2.00
         38.00          2.00          1.00       8103.08       -426.39          2.00
         39.00          3.00          1.00      22026.47      -1157.16          2.00
         40.00          1.00          2.00       8103.08       -426.39          2.00
         41.00          2.00          2.00      22026.47      -1157.16          2.00
         42.00          3.00          2.00      59874.14      -3143.61          2.00
         43.00          1.00          3.00      22026.47      -1157.16          2.00
         44.00          2.00          3.00      59874.14      -3143.61          2.00
         45.00          3.00          3.00     162754.79      -8543.34          2.00
         46.00          1.00          1.00       8914.60       -158.85          1.00
         47.00          2.00          1.00      24232.38       -431.81          1.00
         48.00          3.00          1.00      65870.44      -1173.78          1.00
         49.00          1.00          2.00      24232.38       -431.81          1.00
         50.00          2.00          2.00      65870.44      -1173.78          1.00
         51.00          3.00          2.00     179054.43      -3190.65          1.00
         52.00          1.00          3.00      65870.44      -1173.78          1.00
         53.00          2.00          3.00     179054.43      -3190.65          1.00
         54.00          3.00          3.00     486720.40      -8673.10          1.00
         55.00          1.00          1.00        996.80       -157.55          2.00
         56.00          2.00          1.00       2709.60       -426.39          2.00
         57.00          3.00          1.00       7365.45      -1157.16          2.00
         58.00          1.00          2.00       2709.60       -426.39          2.00
         59.00          2.00          2.00       7365.45      -1157.16          2.00
         60.00          3.00          2.00      20021.36      -3143.61          2.00
```

69	61.00	1.00	3.00	7365.45	-1157.16	2.00
70	62.00	2.00	3.00	20021.36	-3143.61	2.00
71	63.00	3.00	3.00	54423.69	-8543.34	2.00
72	64.00	1.00	1.00	2980.96	-157.55	2.00
73	65.00	2.00	1.00	8103.08	-426.39	2.00
74	66.00	3.00	1.00	22026.47	-1157.16	2.00
75	67.00	1.00	2.00	8103.08	-426.39	2.00
76	68.00	2.00	2.00	22026.47	-1157.16	2.00
77	69.00	3.00	2.00	59874.14	-3143.61	2.00
78	70.00	1.00	3.00	22026.47	-1157.16	2.00
79	71.00	2.00	3.00	59874.14	-3143.61	2.00
80	72.00	3.00	3.00	162754.79	-8543.34	2.00
81	73.00	1.00	1.00	8914.60	-157.76	1.00
82	74.00	2.00	1.00	24232.38	-430.71	1.00
83	75.00	3.00	1.00	65870.44	-1172.68	1.00
84	76.00	1.00	2.00	24232.38	-430.71	1.00
85	77.00	2.00	2.00	65870.44	-1172.68	1.00
86	78.00	3.00	2.00	179054.43	-3189.56	1.00
87	79.00	1.00	3.00	65870.44	-1172.68	1.00
88	80.00	2.00	3.00	179054.43	-3189.56	1.00
89	81.00	3.00	3.00	486720.40	-8672.00	1.00
90	82.00	1.00	1.00	996.80	-156.46	2.00
91	83.00	2.00	1.00	2709.60	-425.29	2.00
92	84.00	3.00	1.00	7365.45	-1156.07	2.00
93	85.00	1.00	2.00	2709.60	-425.29	2.00
94	86.00	2.00	2.00	7365.45	-1156.07	2.00
95	87.00	3.00	2.00	20021.36	-3142.52	2.00
96	88.00	1.00	3.00	7365.45	-1156.07	2.00
97	89.00	2.00	3.00	20021.36	-3142.52	2.00
98	90.00	3.00	3.00	54423.69	-8542.25	2.00
99	91.00	1.00	1.00	2980.96	-156.46	2.00
100	92.00	2.00	1.00	8103.08	-425.29	2.00
101	93.00	3.00	1.00	22026.47	-1156.07	2.00
102	94.00	1.00	2.00	8103.08	-425.29	2.00
103	95.00	2.00	2.00	22026.47	-1156.07	2.00
104	96.00	3.00	2.00	59874.14	-3142.52	2.00
105	97.00	1.00	3.00	22026.47	-1156.07	2.00
106	98.00	2.00	3.00	59874.14	-3142.52	2.00
107	99.00	3.00	3.00	162754.79	-8542.25	2.00
108	100.00	1.00	1.00	8914.60	-159.95	1.00
109	101.00	2.00	1.00	24232.38	-432.90	1.00
110	102.00	3.00	1.00	65870.44	-1174.87	1.00
111	103.00	1.00	2.00	24232.38	-432.90	1.00
112	104.00	2.00	2.00	65870.44	-1174.87	1.00
113	105.00	3.00	2.00	179054.43	-3191.75	1.00
114	106.00	1.00	3.00	65870.44	-1174.87	1.00
115	107.00	2.00	3.00	179054.43	-3191.75	1.00
116	108.00	3.00	3.00	486720.40	-8674.19	1.00
117	109.00	1.00	1.00	996.80	-156.46	2.00
118	110.00	2.00	1.00	2709.60	-425.29	2.00
119	111.00	3.00	1.00	7365.45	-1156.07	2.00
120	112.00	1.00	2.00	2709.60	-425.29	2.00
121	113.00	2.00	2.00	7365.45	-1156.07	2.00
122	114.00	3.00	2.00	20021.36	-3142.52	2.00
123	115.00	1.00	3.00	7365.45	-1156.07	2.00
124	116.00	2.00	3.00	20021.36	-3142.52	2.00
125	117.00	3.00	3.00	54423.69	-8542.25	2.00
126	118.00	1.00	1.00	2980.96	-156.46	2.00
127	119.00	2.00	1.00	8103.08	-425.29	2.00
128	120.00	3.00	1.00	22026.47	-1156.07	2.00
129	121.00	1.00	2.00	8103.08	-425.29	2.00
130	122.00	2.00	2.00	22026.47	-1156.07	2.00
131	123.00	3.00	2.00	59874.14	-3142.52	2.00
132	124.00	1.00	3.00	22026.47	-1156.07	2.00
133	125.00	2.00	3.00	59874.14	-3142.52	2.00

134	126.00	3.00	3.00	162754.79	-8542.25	2.00
135	127.00	1.00	1.00	8914.60	-158.85	1.00
136	128.00	2.00	1.00	24232.38	-431.81	1.00
137	129.00	3.00	1.00	65870.44	-1173.78	1.00
138	130.00	1.00	2.00	24232.38	-431.81	1.00
139	131.00	2.00	2.00	65870.44	-1173.78	1.00
140	132.00	3.00	2.00	179054.43	-3190.65	1.00
141	133.00	1.00	3.00	65870.44	-1173.78	1.00
142	134.00	2.00	3.00	179054.43	-3190.65	1.00
143	135.00	3.00	3.00	486720.40	-8673.10	1.00
144	136.00	1.00	1.00	996.80	-156.46	2.00
145	137.00	2.00	1.00	2709.60	-425.29	2.00
146	138.00	3.00	1.00	7365.45	-1156.07	2.00
147	139.00	1.00	2.00	2709.60	-425.29	2.00
148	140.00	2.00	2.00	7365.45	-1156.07	2.00
149	141.00	3.00	2.00	20021.36	-3142.52	2.00
150	142.00	1.00	3.00	7365.45	-1156.07	2.00
151	143.00	2.00	3.00	20021.36	-3142.52	2.00
152	144.00	3.00	3.00	54423.69	-8542.25	2.00
153	145.00	1.00	1.00	2980.96	-156.46	2.00
154	146.00	2.00	1.00	8103.08	-425.29	2.00
155	147.00	3.00	1.00	22026.47	-1156.07	2.00
156	148.00	1.00	2.00	8103.08	-425.29	2.00
157	149.00	2.00	2.00	22026.47	-1156.07	2.00
158	150.00	3.00	2.00	59874.14	-3142.52	2.00
159	151.00	1.00	3.00	22026.47	-1156.07	2.00
160	152.00	2.00	3.00	59874.14	-3142.52	2.00
161	153.00	3.00	3.00	162754.79	-8542.25	2.00
162	154.00	1.00	1.00	8914.60	-157.76	1.00
163	155.00	2.00	1.00	24232.38	-430.71	1.00
164	156.00	3.00	1.00	65870.44	-1172.68	1.00
165	157.00	1.00	2.00	24232.38	-430.71	1.00
166	158.00	2.00	2.00	65870.44	-1172.68	1.00
167	159.00	3.00	2.00	179054.43	-3189.56	1.00
168	160.00	1.00	3.00	65870.44	-1172.68	1.00
169	161.00	2.00	3.00	179054.43	-3189.56	1.00
170	162.00	3.00	3.00	486720.40	-8672.00	1.00
171	163.00	1.00	1.00	996.80	-155.36	2.00
172	164.00	2.00	1.00	2709.60	-424.20	2.00
173	165.00	3.00	1.00	7365.45	-1154.97	2.00
174	166.00	1.00	2.00	2709.60	-424.20	2.00
175	167.00	2.00	2.00	7365.45	-1154.97	2.00
176	168.00	3.00	2.00	20021.36	-3141.42	2.00
177	169.00	1.00	3.00	7365.45	-1154.97	2.00
178	170.00	2.00	3.00	20021.36	-3141.42	2.00
179	171.00	3.00	3.00	54423.69	-8541.15	2.00
180	172.00	1.00	1.00	2980.96	-155.36	2.00
181	173.00	2.00	1.00	8103.08	-424.20	2.00
182	174.00	3.00	1.00	22026.47	-1154.97	2.00
183	175.00	1.00	2.00	8103.08	-424.20	2.00
184	176.00	2.00	2.00	22026.47	-1154.97	2.00
185	177.00	3.00	2.00	59874.14	-3141.42	2.00
186	178.00	1.00	3.00	22026.47	-1154.97	2.00
187	179.00	2.00	3.00	59874.14	-3141.42	2.00
188	180.00	3.00	3.00	162754.79	-8541.15	2.00
189	181.00	1.00	1.00	8914.60	-159.95	1.00
190	182.00	2.00	1.00	24232.38	-432.90	1.00
191	183.00	3.00	1.00	65870.44	-1174.87	1.00
192	184.00	1.00	2.00	24232.38	-432.90	1.00
193	185.00	2.00	2.00	65870.44	-1174.87	1.00
194	186.00	3.00	2.00	179054.43	-3191.75	1.00
195	187.00	1.00	3.00	65870.44	-1174.87	1.00
196	188.00	2.00	3.00	179054.43	-3191.75	1.00
197	189.00	3.00	3.00	486720.40	-8674.19	1.00
198	190.00	1.00	1.00	996.80	-155.36	2.00

```
       191.00         2.00         1.00      2709.60      -424.20         2.00
       192.00         3.00         1.00      7365.45     -1154.97         2.00
       193.00         1.00         2.00      2709.60      -424.20         2.00
       194.00         2.00         2.00      7365.45     -1154.97         2.00
       195.00         3.00         2.00     20021.36     -3141.42         2.00
       196.00         1.00         3.00      7365.45     -1154.97         2.00
       197.00         2.00         3.00     20021.36     -3141.42         2.00
       198.00         3.00         3.00     54423.69     -8541.15         2.00
       199.00         1.00         1.00      2980.96      -155.36         2.00
       200.00         2.00         1.00      8103.08      -424.20         2.00
       201.00         3.00         1.00     22026.47     -1154.97         2.00
       202.00         1.00         2.00      8103.08      -424.20         2.00
       203.00         2.00         2.00     22026.47     -1154.97         2.00
       204.00         3.00         2.00     59874.14     -3141.42         2.00
       205.00         1.00         3.00     22026.47     -1154.97         2.00
       206.00         2.00         3.00     59874.14     -3141.42         2.00
       207.00         3.00         3.00    162754.79     -8541.15         2.00
       208.00         1.00         1.00      8914.60      -158.85         1.00
       209.00         2.00         1.00     24232.38      -431.81         1.00
       210.00         3.00         1.00     65870.44     -1173.78         1.00
       211.00         1.00         2.00     24232.38      -431.81         1.00
       212.00         2.00         2.00     65870.44     -1173.78         1.00
       213.00         3.00         2.00    179054.43     -3190.65         1.00
       214.00         1.00         3.00     65870.44     -1173.78         1.00
       215.00         2.00         3.00    179054.43     -3190.65         1.00
       216.00         3.00         3.00    486720.40     -8673.10         1.00
       217.00         1.00         1.00       996.80      -155.36         2.00
       218.00         2.00         1.00      2709.60      -424.20         2.00
       219.00         3.00         1.00      7365.45     -1154.97         2.00
       220.00         1.00         2.00      2709.60      -424.20         2.00
       221.00         2.00         2.00      7365.45     -1154.97         2.00
       222.00         3.00         2.00     20021.36     -3141.42         2.00
       223.00         1.00         3.00      7365.45     -1154.97         2.00
       224.00         2.00         3.00     20021.36     -3141.42         2.00
       225.00         3.00         3.00     54423.69     -8541.15         2.00
       226.00         1.00         1.00      2980.96      -155.36         2.00
       227.00         2.00         1.00      8103.08      -424.20         2.00
       228.00         3.00         1.00     22026.47     -1154.97         2.00
       229.00         1.00         2.00      8103.08      -424.20         2.00
       230.00         2.00         2.00     22026.47     -1154.97         2.00
       231.00         3.00         2.00     59874.14     -3141.42         2.00
       232.00         1.00         3.00     22026.47     -1154.97         2.00
       233.00         2.00         3.00     59874.14     -3141.42         2.00
       234.00         3.00         3.00    162754.79     -8541.15         2.00
       235.00         1.00         1.00      8914.60      -157.76         1.00
       236.00         2.00         1.00     24232.38      -430.71         1.00
       237.00         3.00         1.00     65870.44     -1172.68         1.00
       238.00         1.00         2.00     24232.38      -430.71         1.00
       239.00         2.00         2.00     65870.44     -1172.68         1.00
       240.00         3.00         2.00    179054.43     -3189.56         1.00
       241.00         1.00         3.00     65870.44     -1172.68         1.00
       242.00         2.00         3.00    179054.43     -3189.56         1.00
       243.00         3.00         3.00    486720.40     -8672.00         1.00


ans =

          0.00        42.00


comptime =

          0.31

```

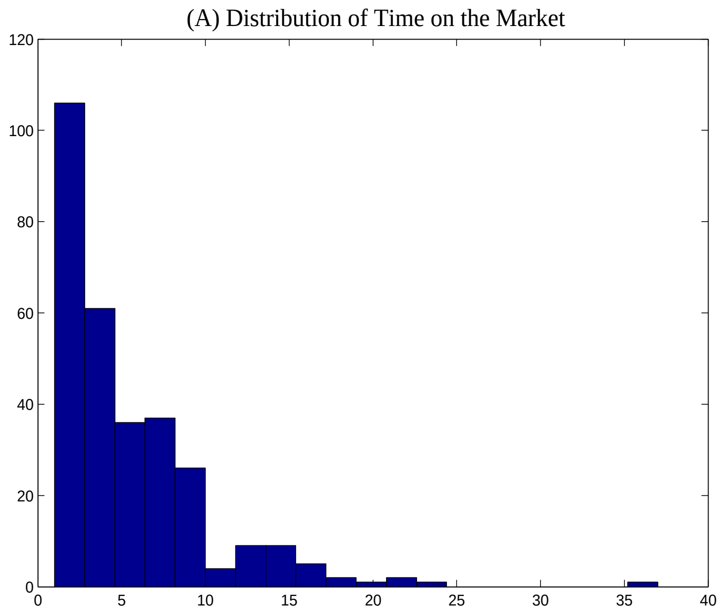
In order to simulate the model, we will need to compute the reservation "price" for each set of attributes. The matlab file `real2.m` computes the reservation prices from the output of file `real1.m`. The file also randomly draw housing shocks, $\varepsilon_{i,t}$, for each house until a reservation wage is reached. The period in which it is sold is recorded as well as its sale price.

```
%
% real2.m is an example matlab program
% to read in the solution to a simple real estate search model
% and compute reservation price for each type of house
% and simulate model
%

format bank

% ----- save output to file ----- %

clear all
diary real2.mout
diary off
delete real2.mout
diary real2.mout

% ----- display date and time of computation ----- %

%format short
date
time0 = clock;

% ----- set control vars ----- %

iters  = 700;

% ----- load data ----- %

X = load('read1.dat')

% ----- compute reservation prices ----- %

P11=min(X(X(:,9)==1 & X(:,2)==1 & X(:,3)==1,7))
P12=min(X(X(:,9)==1 & X(:,2)==1 & X(:,3)==2,7))
P13=min(X(X(:,9)==1 & X(:,2)==1 & X(:,3)==3,7))

P21=min(X(X(:,9)==1 & X(:,2)==2 & X(:,3)==1,7))
P22=min(X(X(:,9)==1 & X(:,2)==2 & X(:,3)==2,7))
P23=min(X(X(:,9)==1 & X(:,2)==2 & X(:,3)==3,7))

P31=min(X(X(:,9)==1 & X(:,2)==3 & X(:,3)==1,7))
P32=min(X(X(:,9)==1 & X(:,2)==3 & X(:,3)==2,7))
P33=min(X(X(:,9)==1 & X(:,2)==3 & X(:,3)==3,7))

P = [P11 P12 P13;
     P21 P22 P23;
     P31 P32 P33 ] ;
```

```

% ----- periods and people ----- %

t = 50
n = 300

% ----- set parameters ----- %

beta0 = 6;
beta1 = 1;
beta2 = 1;

% ----- for each person randomly draw housing hedonic values ----- %

omega = discreteinvrnd([.4,.4,.20],n,2) ;

% ----- for every person, draw a pricing shock
%          and compute a price for each period
%           till the reservation price is meet ----- %

[ep,eppr]=qnwnorm(3,0,.40);


for i = 1:n
 for j = 1:t
  ss           = ep(discreteinvrnd(eppr,1,1));
  price(i,1)   = exp(beta0 + beta1*omega(i,1) + beta2*omega(i,2) + ss ) ;
  exprice(i,1) = exp(beta0 + beta1*omega(i,1) + beta2*omega(i,2) ) ;
  timemkt(i,1) = j;
   if price(i,1) >= floor(P(omega(i,1),omega(i,2))), break, end
 end
end

% ----- plot time on the market histogram ----- %

figure(1)
hist(timemkt,20)
title('(A) Distribution of Time on the Market','fontsize',16,'fontname','times')
print -depsc2 -tiff -r300 -adobecset real2a

% ----- compute mean mean time on the market ----- %

mean(timemkt)

% ----- compute regression coefficient ----- %

inv([ones(n,1) omega]'*[ones(n,1) omega])*[ones(n,1) omega]'*log(price)



% ----- save data ----- %


X = [ log(price) omega(:,1:2) ] ;


save real2.dat X -ascii



comptime = etime(clock, time0)
diary off
```

Figure 2 shows the distribution of time on the market for our simulated economy. The mean time on the market is 5.76 months.

Figure 2: Real Estate Model Simulation



Question 20: *Using the sales prices and housing attributes from the simulated economy, run a regression of the logged sales price on a constant, number of baths, and square footage of the house. Do you get back the prediction equation* $\ln P_{i,t} = 6 + 1 \cdot SQ_i + 1 \cdot Bath_i$*? Why or why not?*

Question 21: *Suppose that we wanted to include idea that house offers from a potential buyer do not occur every period. That is, there is a* λ *percent chance that the housing offer would be zero and a* $1 - \lambda$ *percent chance that the offer will be from our hedonic pricing model:* $\ln P_{i,t} = 6 + 1 \cdot SQ_i + 1 \cdot Bath_i + \varepsilon_{i,t}$*. How would you change the setup of the Bellmans? Would the reservation prices change? If so, how?*

9 Rust's Model

9.1 The setup

If the bus company keeps the ith bus running, $a_{i,t} = 1$, then costs are $c(x_t, \theta) = \theta \cdot x_t$ where x_t is the mileage on the bus. If the engine is replaced, $a_{i,t} = 2$, then costs are $\bar{P} - \bar{P} + c(0, \theta) = \bar{P} - \bar{P} + \theta \cdot 0$. The utility of the agent is

$$u_t = \begin{cases} -\theta \cdot x_t + \epsilon_t^{<1>} & a_{i,t} = 1 \\ -\bar{P} + \bar{P} - \theta \cdot 0 + \epsilon_t^{<2>} & a_{i,t} = 2 \end{cases},$$

where $\epsilon_{i,t}^{<1>}$ and $\epsilon_{i,t}^{<2>}$ determine the utility from the choice. These unobserved components are assumed to have unconditional and conditional expectations of zero:

$$E_t \left[\epsilon_{i,t+1}^{<j>} \right] = 0.$$

Question 22: *What are some economic interpretations of $\epsilon_{i,t}^{<1>}$ and $\epsilon_{i,t}^{<2>}$?*

If the bus company decides not to overhaul, then bus mileage will either stay the same with probability of θ_0, increase by one with probability of θ_1, or increase by two with probability of θ_2. If there are at most 90 intervals of 5000 miles in length, the Markov transition probability matrix would look like:

$$\mathbf{P} = \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & 0 & 0 & 0 & 0 & 0 \\ 0 & P_{2,2} & P_{2,3} & P_{2,4} & 0 & 0 & 0 & 0 \\ 0 & 0 & P_{3,3} & P_{3,4} & P_{3,5} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We assume that if the mileage on the bus gets to 450,000 miles then the bus company will have to overhaul.

9.2 The Bellman's

If the company decides not to overhaul (denoted $a_{i,t} = 1$) at the optimal choice $a_{i,t}$ the value of that choice is:

$$V^{<1>}(\Omega_t) = -\theta \cdot x_t + \epsilon_{i,t}^{<1>} + \beta E_t \{V(\Omega_{t+1})\}$$

For the bus company, the state space includes mileage, $\epsilon_{i,t}^{<1>}$, and $\epsilon_{i,t}^{<2>}$; $\Omega_t = \{x_t, \epsilon_{i,t}^{<1>}, \epsilon_{i,t}^{<2>}\}$. If the company decides to overhaul (denoted $a_{i,t} = 2$), the value of that choice is:

$$V^{<2>}(\Omega_t) = -\bar{P} + \bar{P} - \theta \cdot 0 + \epsilon_{i,t}^{<2>} + \beta E_t \{V(\Omega_{t+1})\}$$

For the bus company, the state space includes mileage, $\epsilon_{i,t}^{<1>}$, and $\epsilon_{i,t}^{<2>}$; $\Omega_t = \{x_t, \epsilon_{i,t}^{<1>}, \epsilon_{i,t}^{<2>}\}$.

To solve the model, we could precede as before by discretizing $\epsilon_{i,t}^{<1>}$ and $\epsilon_{i,t}^{<2>}$ to construct a state space. However, Rust develops a key trick that allows us to basically integrate out these unobserved components. Note that we can write out the Bellman's as:

$$V^{<1>}(\Omega_t) - \epsilon_{i,t}^{<1>} = -\theta \cdot x_t + \beta E_t \{V(\Omega_{t+1})\}$$

Additionally, the following statement is true:

$$V^{<1>}(\Omega_t) - \epsilon_{i,t}^{<1>} = -\theta \cdot x_t + \beta E_t \{V(\Omega_{t+1})\} - \beta E_t \left\{\epsilon_{i,t+1}^{<1>}\right\}$$

since $\beta E_t \left\{\epsilon_{i,t+1}^{<1>}\right\} = 0$. Thus,

$$V^{<1>}(\Omega_t) - \epsilon_{i,t}^{<1>} = -\theta \cdot x_t + \beta E_t \left\{V(\Omega_{t+1}) - \epsilon_{i,t+1}^{<1>}\right\}$$

Letting $V^{<1>}(\Omega_t) - \epsilon_{i,t}^{<1>} = \bar{V}^{<1>}(\Omega_t)$ we get:

$$\bar{V}^{<1>}(\Omega_t) = -\theta \cdot x_t + \beta E_t \left\{ \bar{V}(\Omega_{t+1}) \right\}$$

For the same reasoning, we can get:

$$\bar{V}^{<2>}(\Omega_t) = -\bar{P} + \bar{P} - \theta \cdot 0 + \beta E_t \left\{ \bar{V}(\Omega_{t+1}) \right\}$$

Rusts idea is then; instead of solving for $V^{<1>}(\Omega_t)$ and $V^{<2>}(\Omega_t)$, why not solve for $\bar{V}^{<1>}(\Omega_t)$ and $\bar{V}^{<2>}(\Omega_t)$?

Before we can get $\bar{V}^{<1>}(\Omega_t)$ and $\bar{V}^{<2>}(\Omega_t)$ we need to be able to evaluate the expectation $E_t \left\{ \bar{V}(\Omega_{t+1}) \right\}$. Rust shows in his paper, that if the agent chooses $a_{i,t} = 1$ then

$$E_t \left\{ \bar{V}(\Omega_{t+1}) \right\} = \mathbf{P} \left[\gamma + \log \left(\exp(\bar{V}^{<1>}(\Omega_{t+1})) + \exp(\bar{V}^{<2>}(\Omega_{t+1})) \right) \right]$$

where γ is Euler's constant. Alternatively, if the bus company chooses to redo the engine, the above expectation is:

$$E_t \left\{ \bar{V}(\Omega_{t+1}) \right\} = \mathbf{P}(1,:) \left[\gamma + \log \left(\exp(\bar{V}^{<1>}(\Omega_{t+1})) + \exp(\bar{V}^{<2>}(\Omega_{t+1})) \right) \right]$$

where $\mathbf{P}(1,:)$ is just the first row of the $\mathbf{P}$ matrix.

Therefore, we can guess at $\bar{V}^{<1>}$ and $\bar{V}^{<2>}$, and compute a new set of value functions by the Bellmans's

```
%
% rust1.m is an example matlab program
% to compute rust's economy
%

% ----- save output to file ----- %

clear all
diary rust1.mout
```

```
diary off
delete rust1.mout
diary rust1.mout

% ----- display date and time of computation ----- %

format bank
date
time0 = clock;

% ----- set iteration controls ----- %

iters = 190800;

% ---- set state space ---- %

states = 90;

% ------ set parameters ------ %

beta    = 0.9999;
RC      = 11.7270;

theta1  = 4.8259;
theta2  = 0.577216;
theta30 = 0.3010;
theta31 = 0.6884;
theta32 = 1 - theta30 - theta31;

scale = 0.05;
alt1 = RC;

% ----- create transition probability matrix ----- %

P1 = [ theta30 theta31 theta32 ];

P = zeros(states,states);

for j = 1:(states);
 P(j,j) = theta30;
end;

for j = 1:(states-1);
 P(j,j+1) = theta31;
end;

for j = 1:(states-2);
 P(j,j+2) = theta32;
end;

% - Note: set the last probs to add to one - %

P(states,states) = 1;
P(states-1,states) = 1-theta30;

% ----- discretize milelage ----- %

x = zeros(states,1);

for j = 1:(states-1);
 x(j+1,1) = x(j,1) + 1;
end;

% ----- make the last mileage infinite ----- %

```

```
%x(states,1) = inf;
%x(states,1) = 1000000000;

% ----- value function iteration ----- %
% ----- initial matrices ----- %

v  = zeros(states,1);
v1 = zeros(states,1);
v2 = zeros(states,1);

% ----- start iterations ------ %

crit1 = 3;

for j = 1:iters ;
 if crit1 < 0.000001, break, end;

% ----- compute value function ----- %

 for i = 1:states;
  temp   = exp(v1(i,1)) + exp(v2(i,1))+eps   ;
  w(i,1) = theta2 + log(temp) ;
 end;

% ----- compute payoffs ----- %

 for i = 1:states;
  v1(i,1) = -scale*theta1*x(i,1)+beta*P(i,:)*w ;
  v2(i,1) = -RC-scale*theta1*x(1,1)+beta*P(1,:)*w ;
  payoff(i,:)  = [v1(i,1), v2(i,1)]  ;
 end;

 [newv,policy] = max(payoff');

 newv = newv';
 policy = policy';

 iter = j;

% ----- compute criterion ----- %

 crit1 = sqrt((newv-v)'*(newv-v))/(1+sqrt(v'*v));

% ----- update ----- %

 v = newv;

end;

% ----- print solutions ----- %

[v, policy, x],  [crit1, iter]

% ----- compute probabilities of choice ----- %

for i = 1:states;
 PP1(i,1) = exp(v1(i,1))/(exp(v1(i,1))+exp(v2(i,1))) ;
 PP2(i,1) = exp(v2(i,1))/(exp(v1(i,1))+exp(v2(i,1))) ;
end;

[PP1 PP2]

% ----- simulate markov chain with rule ----- %

for j = 1:10
```

```
 sim(1,j) = ceil(38*rand(1,1));
 for i = 1:1000;
  if sim(i,j) < 38 + ceil(2*(rand(1,1)-.5))
   [temp1, temp2] = markov(P,10,sim(i,j),x');
   sim(i+1,j) = temp1(1,2) + 1;
  else
   sim(i+1,j) = 1;
  end;
 end;
end;

% ----- plot data ---- %

figure(1)
plot(sim(1:100,:))
 title('(A) Mileage of Different Buses','fontsize',16,'fontname','times')
 xlabel('Month','fontsize',12,'fontname','times')
 ylabel('Mileage','fontsize',12,'fontname','times')
axis([0 100 0 55])
pause(3)
print -depsc2 -tiff -r300 -adobecset rust1a

% ----- finish ----- %

comptime = etime(clock, time0)

diary off
```

The output shows that the reservation mileage is 39.

```

ans =

11-Sep-2008


ans =

        -32.94         1.00            0
        -33.89         1.00         1.00
        -34.63         1.00         2.00
        -35.22         1.00         3.00
        -35.71         1.00         4.00
        -36.13         1.00         5.00
        -36.50         1.00         6.00
        -36.84         1.00         7.00
        -37.15         1.00         8.00
        -37.45         1.00         9.00
        -37.73         1.00        10.00
        -38.01         1.00        11.00
        -38.27         1.00        12.00
        -38.53         1.00        13.00
        -38.79         1.00        14.00
        -39.04         1.00        15.00
        -39.29         1.00        16.00
        -39.54         1.00        17.00
        -39.79         1.00        18.00
        -40.03         1.00        19.00
        -40.28         1.00        20.00
```

30	-40.52	1.00	21.00
31	-40.76	1.00	22.00
32	-41.01	1.00	23.00
33	-41.25	1.00	24.00
34	-41.49	1.00	25.00
35	-41.73	1.00	26.00
36	-41.98	1.00	27.00
37	-42.22	1.00	28.00
38	-42.46	1.00	29.00
39	-42.70	1.00	30.00
40	-42.94	1.00	31.00
41	-43.18	1.00	32.00
42	-43.42	1.00	33.00
43	-43.67	1.00	34.00
44	-43.91	1.00	35.00
45	-44.15	1.00	36.00
46	-44.39	1.00	37.00
47	-44.63	1.00	38.00
48	-44.66	2.00	39.00
49	-44.66	2.00	40.00
50	-44.66	2.00	41.00
51	-44.66	2.00	42.00
52	-44.66	2.00	43.00
53	-44.66	2.00	44.00
54	-44.66	2.00	45.00
55	-44.66	2.00	46.00
56	-44.66	2.00	47.00
57	-44.66	2.00	48.00
58	-44.66	2.00	49.00
59	-44.66	2.00	50.00
60	-44.66	2.00	51.00
61	-44.66	2.00	52.00
62	-44.66	2.00	53.00
63	-44.66	2.00	54.00
64	-44.66	2.00	55.00
65	-44.66	2.00	56.00
66	-44.66	2.00	57.00
67	-44.66	2.00	58.00
68	-44.66	2.00	59.00
69	-44.66	2.00	60.00
70	-44.66	2.00	61.00
71	-44.66	2.00	62.00
72	-44.66	2.00	63.00
73	-44.66	2.00	64.00
74	-44.66	2.00	65.00
75	-44.66	2.00	66.00
76	-44.66	2.00	67.00
77	-44.66	2.00	68.00
78	-44.66	2.00	69.00
79	-44.66	2.00	70.00
80	-44.66	2.00	71.00
81	-44.66	2.00	72.00
82	-44.66	2.00	73.00
83	-44.66	2.00	74.00
84	-44.66	2.00	75.00
85	-44.66	2.00	76.00
86	-44.66	2.00	77.00
87	-44.66	2.00	78.00
88	-44.66	2.00	79.00
89	-44.66	2.00	80.00
90	-44.66	2.00	81.00
91	-44.66	2.00	82.00
92	-44.66	2.00	83.00
93	-44.66	2.00	84.00
94	-44.66	2.00	85.00

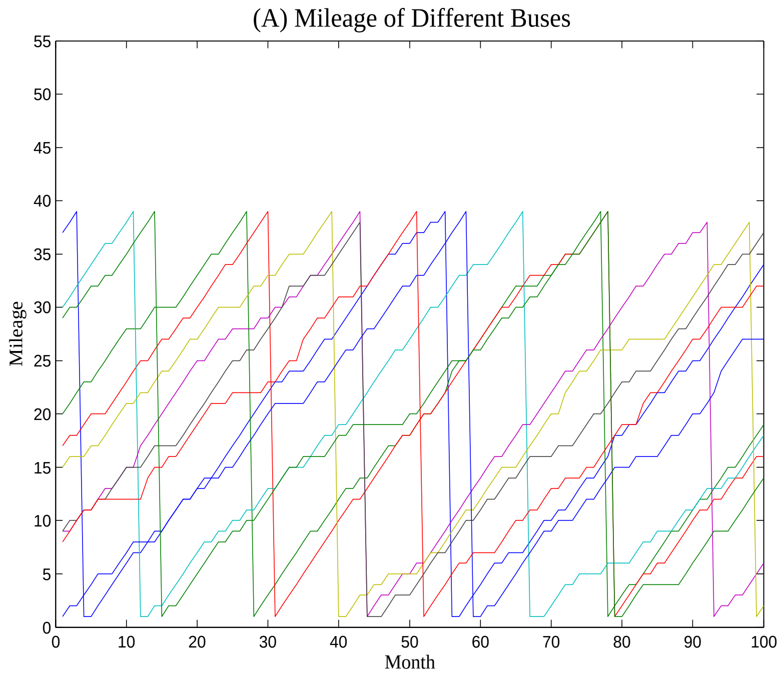
```
         -44.66          2.00         86.00
         -44.66          2.00         87.00
         -44.66          2.00         88.00
         -44.66          2.00         89.00


ans =

           0.00         49.00


ans =

           1.00          0.00
           1.00          0.00
           1.00          0.00
           1.00          0.00
           1.00          0.00
           1.00          0.00
           1.00          0.00
           1.00          0.00
           1.00          0.00
           1.00          0.00
           1.00          0.00
           1.00          0.00
           1.00          0.00
           1.00          0.00
           1.00          0.00
           1.00          0.00
           1.00          0.00
           0.99          0.01
           0.99          0.01
           0.99          0.01
           0.99          0.01
           0.98          0.02
           0.98          0.02
           0.97          0.03
           0.97          0.03
           0.96          0.04
           0.95          0.05
           0.94          0.06
           0.92          0.08
           0.90          0.10
           0.88          0.12
           0.85          0.15
           0.81          0.19
           0.78          0.22
           0.73          0.27
           0.68          0.32
           0.63          0.37
           0.57          0.43
           0.51          0.49
           0.45          0.55
           0.39          0.61
           0.33          0.67
           0.28          0.72
           0.24          0.76
           0.20          0.80
           0.16          0.84
           0.13          0.87
           0.11          0.89
           0.08          0.92
           0.07          0.93
           0.05          0.95
           0.04          0.96
```

```
            0.03          0.97
            0.03          0.97
            0.02          0.98
            0.02          0.98
            0.01          0.99
            0.01          0.99
            0.01          0.99
            0.01          0.99
            0.01          0.99
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00
            0.00          1.00


comptime =

           27.92

```

Question 23: *Solve the real estate model by using Rust's method to integrate out* $\epsilon_{i,t}^{<1>}$ *and* $\epsilon_{i,t}^{<2>}$*? The olny states should be* $SQ_i = [1, 2, 3]'$, $Bath_i = [1, 2, 3]'$, $\varepsilon_{i,t} = [-1.0954, 0, 1.0954]'$.

Simulating the model gives the following graph:

Figure 3: Rust Model Simulation



10 Bayesian Maximum Likelihood

More later . . . . . .