Dynamic Programming Models with Growth

Stuart J. Fowler*
Middle Tennessee State University

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*Economics and Finance Department, Box 27, Middle Tennessee State University, Murfreesboro, TN 37132, Tel: 615-898-2383, Fax: 615-898-5596, email: sfowler@mtsu.edu.

1 Growing Economies

Growth is easily adapted into these models. For example, consider the sequential non-stochastic planning problem:

$$\max_{\{c_t,\ell_t,k_{t+1}\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta u(c_t, \ell_t) \right\}$$

subject to:

$$\begin{aligned} c_t + k_{t+1} &= k_t^{\alpha}(\theta_t n_t)^{1-\alpha} + (1-\delta)k_t, \\ \ell_t + n_t &= 1. \end{aligned}$$

Here, ℓ_t is leisure and n_t are hours devoted to work where the second budget constraint indicates that leisure and work hours have been normalized to one. Note that these are choice variables of the agent and are thus controls.

The variable θ_t represents labor productivity which is increasing each period by a factor of $\bar{\theta}$. That is, the variable θ_t is still assumed to be deterministic but growing: $\theta_t = (\bar{\theta})^t$, where $\bar{\theta} > 1$. Logic suggest that if total income is growing, $\uparrow y_t = k_t^{\alpha}(\uparrow \theta_t n_t)^{1-\alpha}$, then c_t, k_{t+1}, and k_t should as well:

$$\uparrow c_t + \uparrow k_{t+1} = \uparrow k_t^{\alpha}(\uparrow \theta_t n_t)^{1-\alpha} + (1-\delta) \uparrow k_t$$

Because expected discounted utility becomes unbounded in this case, we need to modify our approach for describing the agent's problem.

The best way to recast the problem as a stationary bounded problem is to solve a *scaled version.* Once the scaled version is solved, then the solutions may be unscaled for the actual solutions. How to scale? Well, it typically depends on ones assumptions for preferences and the production process. For example, suppose When utility is of the form:

$$u(c_t, 1-n_t) = \log c_t + \psi \frac{(1-n_t)^{1-\sigma}}{1-\sigma}$$

then it is easy to show that the Eulers' are:

$$-\psi(1-n_t)^{-\sigma} + c_t^{-1}(1-\alpha)k_t^{\alpha}(\theta_t n_t)^{-\alpha}\theta_t \;=\; 0 \tag{1a}$$

$$c_t^{-1} - \beta c_{t+1}^{-1}[\alpha k_{t+1}^{\alpha-1}(\theta_{t+1}n_{t+1})^{1-\alpha} + (1-\delta)] \;=\; 0 \tag{1b}$$

with the budget of $c_t = k_t^{\alpha}(\theta_t n_t)^{1-\alpha} + (1-\delta)k_t - k_{t+1}$.

Question 1: *Show all the steps to derive equations (1a) and (1b).*

Then we ask which variables have trends? Think about hours working; if it had a trend what would happen at n_∞? It wouldn't make sense for hours to have a trend because there are only so many hours in a time period. This leaves us with a good idea of what variables should have trends by examination of the Eulers. Because the intratemporal Euler must hold:

$$\psi(1-n_t)^{-\sigma} = c_t^{-1}(1-\alpha)k_t^{\alpha}(\theta_t n_t)^{-\alpha}\theta_t$$

we see that $c_t^{-1}(1-\alpha)k_t^{\alpha}(\theta_t n_t)^{-\alpha}\theta_t$ must also be stationary. What if we wrote the Euler as

$$\begin{aligned}
\psi(1-n_t)^{-\sigma} &= \frac{\theta_t}{c_t}(1-\alpha)\left(\frac{k_t}{\theta_t}\right)^{\alpha}(n_t)^{-\alpha} \\
&= \frac{\theta_t}{c_t}(1-\alpha)\left(\frac{k_t}{\theta_t}\frac{\theta_{t-1}}{\theta_{t-1}}\right)^{\alpha}(n_t)^{-\alpha} \\
&= \hat{c}_t^{-1}(1-\alpha)\left(\hat{k}_t\frac{1}{\overline{\theta}}\right)^{\alpha}(n_t)^{-\alpha}
\end{aligned}$$

where $\hat{c}_t = c_t/\theta_t$ and $\hat{k}_t = k_t/\theta_{t-1}$. Additionally, redefine the intertemoporal Euler and budget as:

$$
\begin{aligned}
\hat{c}_t^{-1} - \beta \hat{c}_{t+1}^{-1} \bar{\theta}^{-1} [\alpha \left(\hat{k}_{t+1} \frac{1}{\bar{\theta}} \right)^{\alpha-1} (n_{t+1})^{1-\alpha} + (1-\delta)] &= 0 \\
\left(\hat{k}_t \frac{1}{\bar{\theta}} \right)^{\alpha} (n_t)^{1-\alpha} + (1-\delta) \frac{\hat{k}_t}{\bar{\theta}} - \hat{c}_t - \hat{k}_{t+1} &= 0
\end{aligned}
$$

It is now relatively easy to solve the Eulers and budget constraints since they form three equations:

$$
\begin{aligned}
w(\hat{k}_t, \hat{c}_t, n_t) &= 0 \\
v(\hat{c}_t, \hat{k}_{t+1}, \hat{c}_{t+1}, n_{t+1}) &= 0 \\
b(\hat{k}_t, \hat{c}_t, n_t, \hat{k}_{t+1}) &= 0
\end{aligned}
$$

Question 2: *Analytically log-linearize the stationary eulers around their steady states.*

Question 3: *Write a matlab code that solves for the steady states when* $\{\bar{\theta} = 1.05, \alpha = .30, \beta = .99, \psi = 1, \sigma = 2, \delta = 0.025\}$.

2 Growing Stochastic Economies

Consider the sequential stochastic planning problem:

$$
\max_{\{c_t, \ell_t, k_{t+1}\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta u(c_t, \ell_t) \right\}
$$

subject to:

$$
\begin{aligned}
c_t + k_{t+1} &= \lambda_t k_t^{\alpha} (\theta_t n_t)^{1-\alpha} + (1-\delta) k_t, \\
\ell_t + n_t &= 1.
\end{aligned}
$$

Here, ℓ_t is leisure and n_t are hours devoted to work where the second budget constraint indicates that leisure and work hours have been normalized to one. Note that these are choice variables of the agent and are thus controls.

The variable λ_t represents total factor productivity which is assumed to follow an autoregressive process:

$$\log(\lambda_{t+1}) = \phi \log(\lambda_t) + \sigma\varepsilon_{t+1}.$$

where $\varepsilon_{t+1} \sim N(0,1)$. Here, ϕ, and σ are given parameters assumed to be $\phi = 0.90$, and $\sigma = 0.01$.

When utility is of the form:

$$u(c_t, 1-n_t) = \log c_t + \psi \frac{(1-n_t)^{1-\sigma}}{1-\sigma}$$

then it is easy to show that the stationary Eulers' are:

$$\psi(1-n_t)^{-\sigma} = \hat{c}_t^{-1}(1-\alpha)\left(\hat{k}_t\frac{1}{\bar{\theta}}\right)^{\alpha}(n_t)^{-\alpha} \tag{2a}$$

$$\hat{c}_t^{-1} = \beta\hat{c}_{t+1}^{-1}\bar{\theta}^{-1}[\alpha\lambda_{t+1}\left(\hat{k}_{t+1}\frac{1}{\bar{\theta}}\right)^{\alpha-1}(n_{t+1})^{1-\alpha} + (1-\delta)] \tag{2b}$$

where $\hat{c}_t = c_t/\theta_t$, $\hat{k}_t = k_t/\theta_{t-1}$, and with a stationary budget of $\hat{c}_t = \lambda_t\left(\hat{k}_t\frac{1}{\bar{\theta}}\right)^{\alpha}(n_t)^{1-\alpha} + (1-\delta)\frac{\hat{k}_t}{\bar{\theta}} - \hat{k}_{t+1}$.

Question 4: *Show all the steps to derive equations (2a) and (2b).*

Question 5: *Analytically log-linearize the stationary eulers around their steady states.*

Question 6: *Using the above calibrations and Klein's code found on our web page, solve for the stationary solution for this problem.*

3 Exogenously Growing Prices

3.1 The Setup

I now describe all agents in the Walrasian small open economy. The agents include the households, firms, and the home country government. The households make consumption, import consumption, labor, and savings decisions in order to maximize

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(C_t, IM_t, T - L_t) \right\}$$

where E denotes the expectations operator, C_t denotes consumption, IM_t denotes import consumption, L_t denotes labor, and $0 < \beta < 1$ is the households' subjective discount rate. For this paper I use the functional form $u(\cdot) = \psi[\theta \log(C_t) + (1-\theta)\log(IM_t)] + (1-\psi)(T - L_t)$. In each period the households' budget constraint specifies that its uses and sources of consumption be equated. The budget constraint is given by

$$C_t + (1 + \tau_t) pe_t IM_t + I_t = W_t L_t + R_t K_t + TR_t$$

where K_t is capital, TR_t is lump sum transfer from the home country government, investment is defined as $I_t \equiv K_{t+1} - (1-\delta)K_t$, and pe_t is terms of trade. I assume that the terms of trade is determined by the world market and taken as given by the small country. Additionally, $R_t K_t$ denotes interest earnings this period, $W_t L_t$ is labor income, τ_t is tax on imports, and δ is the depreciation rate of capital. The terms of trade is modelled as an exogenous growing process given by

$$pe_t = \overline{pe}^t.$$

In each period, firms combine current labor, H_t, with beginning capital stock, K_t, to produce output, Y_t, through a constant returns to scale production function denoted $f(\cdot)$.

That is, technology is described by

$$\begin{aligned} Y_t &= f(K_t, H_t, A_t) \\ &= A_t K_t^{\alpha} H_t^{(1-\alpha)} \end{aligned}$$

where A_t denotes a stochastic level of technology, $0 < \alpha < 1$. The level of technology evolves according to the following process:

$$\log(A_t) = \rho_A \log(A_{t-1}) + \varepsilon_{A,t}.$$

The firms, which are assumed to be owned by the households, maximize profits in each period t. Profits are defined as total receipts, Y_t, minus total outlays, $W_t H_t + R_t K_t$. Thus, the firms maximize

$$\pi_t = Y_t - W_t H_t - R_t K_t.$$

With these specifications the wage rate and rental rate of capital are given by

$$\begin{aligned} w_t &= f_l(t) \\ R_t &= f_k(t) \end{aligned}$$

where $f_x \equiv \partial f / \partial x$.

The home country government's budget constraint is

$$TR_t = \tau_t pe_t IM_t,$$

The tax rate is assumed to be constant:

$$\tau_t = \bar{\tau}.$$

For the tax regime, the tax level is set so the import-output ratio is roughly 30.0%.

3.2 Equilibrium

In equilibrium, households maximize utility, firms maximize profits, and all markets clear. Clearing of the foreign trade market requires that the value of imports and the value of exports plus government foreign transfers be equated,

$$X_t = pe_t IM_t.$$

Clearing of the goods market requires

$$Y_t = C_t + X_t + I_t.$$

Clearing of the labor market requires

$$L_t = H_t.$$

Using the market clearing conditions and the optimizing behavior of the agents, three efficiency conditions can be obtained. The basic premise of efficiency conditions is that the costs and benefits must equate when agents deviate from their optimal plans. First, if households were to increase labor by one unit, the utility cost would be $-u_l(t)$, where $u_l \equiv \partial u/\partial L$. The discounted benefit of this action to the household would be $u_c(t)f_l(t)$. In equilibrium, the costs and benefits of this action must equate. Thus,

$$-u_l(t) = u_c(t)f_l(t).$$

Similarly, if households were to decrease consumption today by one unit the cost of this action would decrease utility by $u_c(t)$. On the benefit side, the extra unit of consumption would generate expected discounted $f_k(t+1)+(1-\delta)$. If the original plan is optimal, as we suppose, then these costs and benefits must be equal:

$$u_c(t) = \beta E_t \left[u_c(t+1) \left(f_k(t+1) + (1-\delta) \right) \right].$$

The third and final efficiency condition is given by:

$$u_c(t) = \frac{u_{im}(t)}{pe_t(1+\tau_t)}.$$

Question 7: *Argue for a scaling method that will transform this economy into a stationary version.*

3.3 Results

All calibration for this paper is from Feve and Langot. Table 1 presents the calibrated parameters.

$\beta = 0.99$	$\theta = 0.636$
$\delta = 0.0117$	$\psi = 0.99562$
$\alpha = 0.2891$	$T = 1428$
$\bar{pe} = 1.02$	$\rho_A = 0.9862$

Table 1: Parameters

Question 8: *Show all the steps to derive the equilibrium equations.*

Question 9: *Write a matlab code that solves for the steady states.*

Question 10: *Analytically log-linearize the stationary eulers around their steady states.*

Question 11: *Using the above calibrations and Klein's code found on our web page, solve for the stationary solution for this problem.*