The Distribution of Ouput and Investment

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1 The Setup

Our method finds a linear solution for the evolution of capital and the rules for consumption and labor hours. They are of the form:

$$\begin{aligned} \mathbf{x}_{t+1} &= \mathbf{F}\mathbf{x}_t + \mathbf{C}\varepsilon_{t+1} \\ \mathbf{X}_t &= \mathbf{H}\mathbf{x}_t \end{aligned}$$

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where

$$\mathbf{x}_t = \begin{bmatrix} \log(\lambda_t/\bar{\lambda}) \\ \log(k_t/\bar{k}) \end{bmatrix}$$

$$\mathbf{X}_t = \begin{bmatrix} \log(c_t/\bar{c}) \\ \log(n_t/\bar{n}) \end{bmatrix}$$

Question 1: *Show how to write* $\log(y_t/\bar{y})$ *as a function of* $\mathbf{x}_t$.

Recall that $\log(y_t) = \log(\lambda_t) + \alpha\log(k_t) + (1-\alpha)\log(n_t)$. Thus, the following must be true: $\log(\bar{y}) = \log(\bar{\lambda}) + \alpha\log(\bar{k}) + (1-\alpha)\log(\bar{n})$. Subtracting the two gives:

$$\log(y_t/\bar{y}) = \log(\lambda_t/\bar{\lambda}) + \alpha\log(k_t/\bar{k}) + (1-\alpha)\log(n_t/\bar{n})$$

In matrices, the above can be written as:

$$\log(y_t/\bar{y}) = [1, \alpha]\mathbf{x}_t + [0, (1-\alpha)]\mathbf{X}_t$$

Continuing gives:

$$\begin{aligned} \log(y_t/\bar{y}) &= [1, \alpha]\mathbf{x}_t + [0, (1-\alpha)]\mathbf{H}\mathbf{x}_t \\ &= [[1, \alpha] + [0, (1-\alpha)]\mathbf{H}]\,\mathbf{x}_t \end{aligned}$$

Question 2: *Show how the augment* $\mathbf{X}_t$ *and* $\mathbf{H}$ *to include* $\log(y_t/\bar{y})$.

Let

$$\mathbf{X}_t^{<1>} = \begin{bmatrix} \log(c_t/\bar{c}) \\ \log(n_t/\bar{n}) \\ \log(y_t/\bar{y}) \end{bmatrix},$$

and

$$\mathbf{H}^{<1>} = \begin{bmatrix} \mathbf{H} \\ [[1, \alpha] + [0, (1-\alpha)]\mathbf{H}] \end{bmatrix}.$$

Therefore, $\mathbf{X}_t^{<1>} = \mathbf{H}^{<1>}\mathbf{x}_t$ is the model augmented with output.

Question 3: *Show how to write* $\log(i_t/\bar{\imath})$ *as a function of* $\mathbf{x}_t$*.*

Notice that we have a problem since logged investment is still nonlinear:

$$\log(i_t) = \log\left(e^{\log(k_{t+1})} + (1-\delta)e^{\log(k_t)}\right).$$

Linearizing around the steady states gives:

$$\begin{aligned} \log(i_t) &= \log\left(e^{\log(\bar{k})} + (1-\delta)e^{\log(\bar{k})}\right) + \frac{\bar{k}}{\bar{k}+(1-\delta)\bar{k}}\log(k_{t+1}/\bar{k}) + \frac{(1-\delta)\bar{k}}{\bar{k}+(1-\delta)\bar{k}}\log(k_t/\bar{k}) \\ &= \log(\bar{\imath}) + \frac{\bar{k}}{\bar{k}+(1-\delta)\bar{k}}\log(k_{t+1}/\bar{k}) + \frac{(1-\delta)\bar{k}}{\bar{k}+(1-\delta)\bar{k}}\log(k_t/\bar{k}) \end{aligned}$$

Or, by subtracting $\log(\bar{\imath})$:

$$\log(i_t/\bar{\imath}) = \frac{\bar{k}}{\bar{k}+(1-\delta)\bar{k}}\log(k_{t+1}/\bar{k}) + \frac{(1-\delta)\bar{k}}{\bar{k}+(1-\delta)\bar{k}}\log(k_t/\bar{k})$$

Notice that we can write this in matrices as:

$$\log(i_t/\bar{\imath}) = [0, \frac{\bar{k}}{\bar{k}+(1-\delta)\bar{k}}]\mathbf{x}_{t+1} + [0, \frac{(1-\delta)\bar{k}}{\bar{k}+(1-\delta)\bar{k}}]\mathbf{x}_t$$

Since $\mathbf{x}_{t+1} = \mathbf{F}\mathbf{x}_t + \mathbf{C}\varepsilon_{t+1}$, we get

$$\log(i_t/\bar{\imath}) = [0, \frac{\bar{k}}{\bar{k}+(1-\delta)\bar{k}}][\mathbf{F}\mathbf{x}_t + \mathbf{C}\varepsilon_{t+1}] + [0, \frac{(1-\delta)\bar{k}}{\bar{k}+(1-\delta)\bar{k}}]\mathbf{x}_t$$

Simplifying gives:

$$\log(i_t/\bar{\imath}) = \left[[0, \frac{\bar{k}}{\bar{k}+(1-\delta)\bar{k}}]\mathbf{F} + [0, \frac{(1-\delta)\bar{k}}{\bar{k}+(1-\delta)\bar{k}}]\right]\mathbf{x}_t + [0, \frac{\bar{k}}{\bar{k}+(1-\delta)\bar{k}}]\mathbf{C}\varepsilon_{t+1}$$

In our case, $\mathbf{C} = \begin{bmatrix} \sigma \\ 0 \end{bmatrix}$, thus $[0, \frac{\bar{k}}{\bar{k}+(1-\delta)\bar{k}}]\begin{bmatrix} \sigma \\ 0 \end{bmatrix} = 0$, therefore:

$$\log(i_t/\bar{\imath}) = \left[[0, \frac{\bar{k}}{\bar{k}+(1-\delta)\bar{k}}]\mathbf{F} + [0, \frac{(1-\delta)\bar{k}}{\bar{k}+(1-\delta)\bar{k}}]\right]\mathbf{x}_t$$

Question 4: *Show how the augment* $\mathbf{X}_t^{<1>}$ *and* $\mathbf{H}^{<1>}$ *to include* $\log(i_t/\bar{\imath})$.

Let

$$\mathbf{X}_t^{<2>} = \begin{bmatrix} \mathbf{X}_t^{<1>} \\ \log(i_t/\bar{\imath}) \end{bmatrix},$$

and

$$\mathbf{H}^{<2>} = \begin{bmatrix} \mathbf{H}^{<1>} \\ [0, \frac{\bar{k}}{\bar{k}+(1-\delta)\bar{k}}]\mathbf{F} + [0, \frac{(1-\delta)\bar{k}}{\bar{k}+(1-\delta)\bar{k}}] \end{bmatrix}.$$

Therefore, $\mathbf{X}_t^{<2>} = \mathbf{H}^{<2>}\mathbf{x}_t$ is the model augmented with investment.

2 Variances

What is the variance/covariance matrix of $\mathbf{X}_t^{<2>}$? Note that $\mathbf{X}_t^{<2>} = \mathbf{H}^{<2>}\mathbf{x}_t$, thus $\mathbf{X}_t^{<2>}\mathbf{X}_t^{<2>\prime} = \mathbf{H}^{<2>}\mathbf{x}_t\mathbf{X}_t^{<2>\prime} = \mathbf{H}^{<2>}\mathbf{x}_t\mathbf{x}_t'\mathbf{H}^{<2>\prime}$. Taking the expectations gives:

$$E[\mathbf{X}_t^{<2>}\mathbf{X}_t^{<2>\prime}] = \mathbf{H}^{<2>}E[\mathbf{x}_t\mathbf{x}_t']\mathbf{H}^{<2>\prime}$$

Also note that $E[\mathbf{x}_t\mathbf{x}_t']$ is the variance/covariance matrix of the states defined by $\mathbf{\Gamma}(0) = dlyap1(\mathbf{F}, \mathbf{CC}')$. Thus, $E[\mathbf{X}_t^{<2>}\mathbf{X}_t^{<2>\prime}] = \mathbf{H}^{<2>}\mathbf{\Gamma}(0)\mathbf{H}^{<2>\prime}$. We can proceed to find the autocovariances by noting that $\mathbf{X}_{t+1}^{<2>} = \mathbf{H}^{<2>}\mathbf{x}_{t+1}$:

$$\begin{aligned} E[\mathbf{X}_{t+1}^{<2>}\mathbf{X}_t^{<2>\prime}] &= \mathbf{H}^{<2>}E[\mathbf{x}_{t+1}\mathbf{x}_t']\mathbf{H}^{<2>\prime} \\ &= \mathbf{H}^{<2>}\mathbf{\Gamma}(1)\mathbf{H}^{<2>\prime} \end{aligned}$$