On the Frame-Stewart Conjecture about the Towers of Hanoi

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Abstract

The multi-peg Towers of Hanoi problem consists of k pegs mounted on a board together with n disks of different sizes. Initially these disks are placed on one peg in order of size, with the largest on the bottom. The rules of the problem allow disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk. The goal of the problem is to transfer all the disks to another peg with the minimum number of moves, denoted $H(n,k)$. An easy recursive argument shows that $H(n,3) = 2^n - 1$. However, the value of $H(n,k)$ is unknown for each $k \geq 4$.

In 1941, Frame and Stewart independently proposed the following algorithm scheme to solve the Tower of Hanoi problem with $k \geq 4$ pegs:

1. Recursively transport a stack of $n - i$ smallest disks from the first peg to a temporary peg, using all k pegs;
2. Transport the remaining stack of i largest disks from the first peg to the final peg, using $k - 1$ peg and ignoring the peg occupied by the smaller disks;
3. Recursively transport the smallest $n-i$ disks from the pemporary peg to the final peg, using all k pegs.

The Frame-Stewart number, denoted $FS(n,k)$, is the minimum number of moves needed to solve the Towers of Hanoi problem using the above Frame-Stewart algorithm scheme. Thus $FS(n,k)$ has the following recursive formula:

$$FS(n,k) = \begin{cases} 2^n - 1 & \text{if } k = 3, \\ \min_{1 \leq i < n}\{2FS(n-i,k) + FS(i,k-1)\} & \text{if } k \geq 4. \end{cases}$$

The Frame-Stewart number $FS(n,k)$ is called the "presumed optimal" solution since no proof has ever been found that an optimal algorithm must be of this scheme. Proving $FS(n,k) = H(n,n)$ has become a notorious open problem. The assumption $FS(n,k) = H(n,n)$ is called the Frame-Stewart conjecture by many people now.

In this talk, we prove that $FS(n,k)$ and $H(n,k)$ both have the same order of magnitude of $2^{(1\pm o(1))(n(k-2)!)^{1/(k-2)}}$. This provides the strongest evidence so far to support the Frame-Stewart conjecture.