The Chain Rule for Derivatives.

THEOREM (*Chain Rule for Derivatives*). If the function k is a composition of two functions:

$$k(x) = f \circ g(x) = f[g(x)],$$

then its derivative can be formed by
 i. differentiating *f* with respect to $g(x)$ and
 ii. multiplying by $g'(x)$:

$$k'(x) = \big[f[g(x)]\big]' = f'[g(x)].\,g'(x)$$

In differential notation:

$$\frac{d(f[g(x)])}{dx} = \frac{d(f[g(x)])}{dg(x)}.\frac{dg(x)}{dx}$$

PROOF. By the definition of the derivative,

$$k'(x) = \lim_{h\to 0}\frac{k(x+h)-k(x)}{h} = \lim_{h\to 0}\frac{f[g(x+h)]-f[g(x)]}{h}$$

Since $g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$, we can write $g(x + h)$ as $g(x+h) = g'(x)h + g(x)$ as $h \to 0$
and so

$$k'(x) = \lim_{h\to 0}\frac{f[g(x)+g'(x)h]-f[g(x)]}{h}$$

Multiplying numerator and denominator by $g'(x)$ and renaming $g'(x)h = h_2$

$$k'(x) = \lim_{h\to 0}\frac{f[g(x)+g'(x)h]-f[g(x)]}{g'(x)h}.\,g'(x)$$

Since $g'(x)$ is a finite number, $g'(x)h = h_2 \to 0$ as $h \to 0$,

$$k'(x) = \underbrace{\lim_{h_2\to 0}\frac{f[g(x)+h_2]-f[g(x)]}{h_2}}_{\text{by definition, } f'[g(x)]}.\,g'(x) = f'[g(x)].\,g'(x)$$ □

EXAMPLE: $k(x) = \sqrt{x^2-1}$ is a composition of the functions $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$,
so its derivative is formed by ignoring the complexities of the argument of *f* :

$f'(x) = \frac{1}{2\sqrt{x}}$ so $f'[g(x)] = \frac{1}{2\sqrt{x^2-1}}$, *then* differentiating $g(x)$:

$$k'(x) = f'[g(x)].\,g'(x) = \frac{1}{2\sqrt{x^2-1}}.(2x) = \frac{x}{\sqrt{x^2-1}}.$$

EXERCISES: Find the derivatives of the following functions with respect to the indicated variable:

1. $f(x) = (x^2-2x-3)^5$. $\frac{df}{dx} =$
2. $g(t) = \frac{3}{t^2+2}$. $\frac{dg}{dt} =$
3. $p(s) = e^{\frac{1}{s}}$. $\frac{dp}{ds} =$
4. $k(x) = \sqrt{\frac{1}{x^2+5}}$. $\frac{dk}{dx} =$
5. $j(t) = \left(\frac{3}{t^2+2}\right)^4$. $\frac{dj}{dt} =$
6. $r(x) = 2^{\sqrt{x}}$. $\frac{dr}{dx} =$
7. $h(t) = \sin(\sqrt{t})$. $\frac{dh}{dt} =$
8. $q(x) = e^{\sin\sqrt{x}}$. $\frac{dq}{dx} =$